Bound state transfer matrix for $AdS_5 \times S^5$ superstring

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ABSTRACT: We apply the algebraic Bethe ansatz technique to compute the eigenvalues of the transfer matrix constructed from the general bound state S-matrix of the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. This allows us to verify certain conjectures on the quantum characteristic function, and to extend them to the general case.

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Contents

1 Introduction

The current challenge in the study of integrability in AdS/CFT [1–15] is to find an exact solution to the planar spectral problem for composite gauge-invariant operators of finite length. The success obtained in the computation of Lüscher's corrections to anomalous dimensions in the framework of the string sigma model [16–22], combined with encouraging indications concerning the applicability of the TBA approach [23–34], strengthens the expectations that such a solution may be reached in the near future.

The various attempts appearing in the literature have partially been based on powerful conjectures obtained by comparison with well known integrable models, and by applying procedures that are well established in standard cases. The confirmations obtained through direct computation have then reinforced these conjectures. However, the AdS/CFT integrable model is in itself very peculiar, and its distinguished features make one wonder how far the analogies with previously studied cases can be stretched. Therefore, it is important to be able to provide an alternative way to derive, from first principles, the relevant system of equations characterizing the solution to the problem.

General criteria of integrability dictate that the complete solution of the finite-size problem can be obtained, once the full set of (bound) states of the asymptotic spectrum, and all their scattering matrices, are known.[1] Explicit fulfilment of this last requirement was missing until very recently, when the complete set of bound state S-matrices has been determined [35]. This opens the possibility to directly evaluate the associated transfer matrix, a quantity which plays an important role in the recent studies of the TBA and Y-system of the AdS/CFT integrable model in [25, 27] and also in [33]. Finding eigenvalues of the transfer matrix corresponding to generic bound state representations is precisely the task we perform in this paper.

The method we will use consists in constructing the monodromy matrix in arbitrary bound state representations, by using the general expression for the S-matrix describing scattering of two bound states, and to diagonalize the corresponding transfer matrix by means of the Algebraic Bethe Ansatz (ABA) technique. As usual, the spectrum is generated by applying certain operator entries of the monodromy matrix to a chosen vacuum. Requiring the corresponding state to be an eigenstate of the transfer matrix results into the full set of eigenvalues and the associated Bethe equations. This program has been already carried out for the transfer matrix with all legs in the fundamental representation [36], and our result generalizes it to arbitrary bound state representations.

Our general result is the formula (3.45) for spectrum of the transfer matrix, to be supplemented with the auxiliary Bethe equations (3.46). When restricting this general formula to the $\mathfrak{su}(2)$ vacuum (symmetric representation defined in the main body of the paper), we obtain the following expression

$$
\Lambda(q|\vec{p}) = 1 + \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathscr{X}_{\ell_0}^{\ell_0,0} \right]
$$
$$
- 2 \sum_{k=0}^{\ell_0-1} \prod_{i=1}^{K^{\mathrm{I}}} \left(\frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \mathscr{X}_k^{k,0} \right) + \sum_{a=\pm} \sum_{k=1}^{\ell_0-1} \prod_{i=1}^{K^{\mathrm{I}}} \lambda_a(q, p_i, k),
$$

where $\mathscr{X}_{\ell_0}^{\ell_0,0}$ is given by (3.11), $\mathscr{X}_k^{k,0}$ by (3.5) and $\lambda_{\pm}$ by (3.20). In the case of fundamental representations in the physical space, we can also make a different choice of the vacuum, and select the $\mathfrak{sl}(2)$ vacuum (which corresponds to antisymmetric representations). The formula we obtain in this case is (4.7), which we also present here for convenience:

$$
\Lambda(q|\vec{p}) = (\ell_0 + 1) \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-}} - \ell_0 \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}} -
$$
$$
- \ell_0 \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} + (\ell_0 - 1) \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}} \sqrt{\frac{x_0^+}{x_0^-}}.
$$

After obtaining the explicit solution, we are in the position of comparing with certain results and conjectures existing in the literature. First of all, we re-obtain the set of bound state Bethe equations derived in [37]. Secondly, by studying the eigenvalues reported above,

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[1]Strictly speaking, this is only known for the ground state.

with all the physical legs in the fundamental representation, we were able to confirm the expression for the transfer matrix proposed by [38], and utilized in [25]. This agreement is found both in the symmetric and in the antisymmetric representation. In [38], the analysis was based on fusion properties and the expansion of a quantum characteristic function. This is an operator used for the construction of transfer matrix eigenvalues in various symmetric representations, in the context of Baxter and Hirota equations [39]. An educated guess for the present situation was made in [38], generalizing the proposal in [40]. See also the papers [41–43], where the Bazhanov-Reshetikhin [44] formula for fusion has been considered in the case of standard $\mathfrak{gl}(m|n)$ superalgebras. Our formulas therefore simultaneously furnish a proof of these conjectured properties in the centrally-extended case, and provide a generalization to arbitrary bound states.

Most importantly, the transfer matrix in the symmetric representation obtained through the fusion procedure [38] exhibits an explicit dependence on the kinematical parameters corresponding to the bound state constituents. The result we find shows that this dependence is, in fact, artificial, since we are able to unambiguously re-expresses the corresponding eigenvalue purely in terms of parameters characterizing the bound state as a whole.

As a check, we have also reproduced the Lüscher corrections found in the literature [17, 19, 25] directly from the transfer matrix eigenvalues. The results we obtain allow one to enlarge the set of operators for which Lüscher's corrections can be explicitly found to those which correspond to bound states of the world-sheet theory.

We point out that we have explicitly carried out the full calculation of the transfer-matrix eigenvalues corresponding only to the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ vacua, to one excitation over the $\mathfrak{su}(2)$ vacuum,[2] and to a certain class of excited states (see appendix C). This, together with the fact that the formulas we deduce perfectly agree with those obtained *via* the expansion of the quantum characteristic function (even with an arbitrary configuration of auxiliary roots) makes us confident about our findings.

The paper is organized as follows. In the next section we recall the necessary facts concerning the construction of the bound state scattering matrix and define the corresponding transfer matrix. In section 3 we use the ABA technique to diagonalize this transfer matrix. We discuss two particular choices of the vacuum, corresponding to the highest weight state of the totally symmetric and anti-symmetric representations of $\mathfrak{su}(2|2)$. Finally, we report in several appendices useful material and computations.

2 Scattering and transfer matrices of the string sigma model

In what follows we will apply the algebraic Bethe ansatz technique to diagonalize the transfer matrix corresponding to bound state representations of the $\text{AdS}_5 \times \text{S}^5$ string sigma model. The emerging Bethe ansatz exhibits a nested structure and, therefore, to keep the discussion transparent, in this section we will fix and explain the notation used throughout the paper. We will also recall some relevant facts about the bound state representations.

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[2]The double-excitation case already involves a number of terms of order one hundred, and its complete treatment would therefore involve a much bigger effort.

2.1 Bound states and their S-matrix

In the uniform light-cone gauge [45] the symmetry algebra of the $\text{AdS}_5 \times \text{S}^5$ string sigma model in the decompactification limit is (two copies of) the centrally extended $\mathfrak{su}(2|2)$ superalgebra [11]. The latter is also a symmetry algebra of the spin chain Hamiltonian associated with $\mathcal{N} = 4$ super Yang-Mills theory [7]. The asymptotic spectrum of the light-cone sigma model consists of elementary particles, i.e. the ones transforming in the fundamental representations of $\mathfrak{su}(2|2)$, and of their bound states. An ℓ-particle bound state transforms in the tensor product of two 4ℓ-dimensional atypical totally symmetric multiplets of $\mathfrak{su}(2|2)$.

The centrally-extended algebra $\mathfrak{su}(2|2)$ consists of bosonic generators $\mathbb{R}, \mathbb{L}$ (generating two copies of $\mathfrak{su}(2)$), supersymmetry generators $\mathbb{Q}, \mathbb{G}$ and central charges $\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger$. The generators $\mathbb{Q}, \mathbb{G}$ are conjugate to each other. For the string sigma model, $\mathbb{H}$ corresponds to the light-cone Hamiltonian and the central charge $\mathbb{C}$ is a function of the world-sheet momentum.

A convenient way to deal with bound state representations and the S-matrix action is to use the superspace formalism [46]. One considers the vector space of analytic functions $\Phi(w, \theta)$ of two bosonic variables $w_{1,2}$ and two fermionic variables $\theta_{3,4}$. Any such function can be expanded as

$$\begin{aligned}
\Phi(w, \theta) &= \sum_{\ell=0}^{\infty} \Phi_\ell(w, \theta), \\
\Phi_\ell &= \phi^{a_1 \ldots a_\ell} w_{a_1} \ldots w_{a_\ell} + \phi^{a_1 \ldots a_{\ell-1} \alpha} w_{a_1} \ldots w_{a_{\ell-1}} \theta_\alpha + \\
&\quad \phi^{a_1 \ldots a_{\ell-2} 34} w_{a_1} \ldots w_{a_{\ell-2}} \theta_3 \theta_4.
\end{aligned} \tag{2.1}$$

Restriction to Φ_ℓ furnishes an atypical totally symmetric representation of dimension 4ℓ. It is realized on a graded vector space with basis $|e_{a_1 \ldots a_\ell}\rangle, |e_{a_1 \ldots a_{\ell-1}\alpha}\rangle, |e_{a_1 \ldots a_{\ell-2}34}\rangle$, where a_i are bosonic indices and α, β are fermionic indices, and each of the basis vectors is totally symmetric in the bosonic indices and anti-symmetric in the fermionic indices. In terms of the above analytic functions, the basis vectors of the totally symmetric representation can be identified as $|e_{a_1 \ldots a_\ell}\rangle \leftrightarrow w_{a_1} \ldots w_{a_\ell}, |e_{a_1 \ldots a_{\ell-1}\alpha}\rangle \leftrightarrow w_{a_1} \ldots w_{a_{\ell-1}} \theta_\alpha$ and $|e_{a_1 \ldots a_{\ell-1}34}\rangle \leftrightarrow w_{a_1} \ldots w_{a_{\ell-2}} \theta_3 \theta_4$, respectively. In the superspace formalism, the algebra generators are represented by differential operators.

Consider two-particle states. We denote the bound state numbers of the scattering particles as ℓ_1 and ℓ_2, respectively. In superspace the tensor product of the corresponding bound state representations is given by

$$\Phi_{\ell_1}(w, \theta) \Phi_{\ell_2}(v, \vartheta), \tag{2.2}$$

where w, θ denote the superspace variables of the first particle and v, ϑ describe the representation of the second particle.

Because of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariance, when acting on such a tensor product representation space, the S-matrix.[3] $\mathbb{S}$ leaves invariant five different subspaces [35]. Each of these subspaces is characterized by a specific assignment of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Dynkin labels, which

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[3]In this picture the S-matrix is understood as an operator $\mathbb{S} : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$.

are quantum numbers that are trivially conserved under scattering. Two pairs of these subspaces are simply related to each other by exchanging the type of fermions appearing, as described below. It leaves only three non-equivalent cases, which we list here below [35].

Case I. The standard basis for this vector space, which we will concisely call V^{I}, is

$$|k,l\rangle^{\mathrm{I}} \equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}}, \tag{2.3}$$

for all $k+l=N$. The range of k,l here and in the cases below is straightforwardly read off from the definition of the states. In particular, k ranges from 0 to ℓ_1-1, and l ranges from 0 to ℓ_2-1. For fixed N, this gives in this case $N+1$ different vectors. We get another copy of Case I if we exchange the index 3 with 4 in the fermionic variable, with the same S-matrix.

Case II. The standard basis for this space V^{II} is

$$\begin{aligned}
|k,l\rangle_1^{\mathrm{II}} &\equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k} \ \underbrace{v_1^{\ell_2-l} v_2^l}, \\
|k,l\rangle_2^{\mathrm{II}} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}, \\
|k,l\rangle_3^{\mathrm{II}} &\equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}, \\
|k,l\rangle_4^{\mathrm{II}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l},
\end{aligned} \tag{2.4}$$

where $k+l=N$ as before.[4] It is easily seen that we get in this case $4N+2$ states. Once again, exchanging 3 with 4 in the fermionic variable gives another copy of Case II, with the same S-matrix.

Case III. For fixed $N = k + l$, the dimension of this vector space V^{III} is $6N$. The standard basis is

$$\begin{aligned}
|k,l\rangle_1^{\mathrm{III}} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k} \ \underbrace{v_1^{\ell_2-l} v_2^l}, \\
|k,l\rangle_2^{\mathrm{III}} &\equiv \underbrace{w_1^{\ell_1-k} w_2^k} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}, \\
|k,l\rangle_3^{\mathrm{III}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}} \ \underbrace{v_1^{\ell_2-l} v_2^l}, \\
|k,l\rangle_4^{\mathrm{III}} &\equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}, \\
|k,l\rangle_5^{\mathrm{III}} &\equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k} \ \underbrace{\vartheta_4 v_1^{\ell_2-l} v_2^{l-1}}, \\
|k,l\rangle_6^{\mathrm{III}} &\equiv \underbrace{\theta_4 w_1^{\ell_1-k} w_2^{k-1}} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}.
\end{aligned} \tag{2.5}$$

As was explained in [35], the different cases are mapped into one another by use of the (opposite) coproducts of the (Yangian) symmetry generators. The S-matrix has the

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[4]We will from now on, with no risk of confusion, omit indicating "Space 1" and "Space 2" under the curly brackets.

following block-diagonal form

$$
\mathbb{S} = \begin{pmatrix} \boxed{\mathscr{X}} & & & & \\ & \boxed{\mathscr{Y}} & & \LARGE 0 & \\ & & \boxed{\mathscr{Z}} & & \\ & \LARGE 0 & & \boxed{\mathscr{Y}} & \\ & & & & \boxed{\mathscr{X}} \end{pmatrix}.
\tag{2.6}
$$

The outer blocks scatter states from V^{I}

$$
\mathscr{X} : V^{\mathrm{I}} \longrightarrow V^{\mathrm{I}},
\tag{2.7}
$$

$$
|k,l\rangle^{\mathrm{I}} \;\mapsto\; \sum_{m=0}^{k+l} \mathscr{X}_m^{k,l} |m, k+l-m\rangle^{\mathrm{I}},
\tag{2.8}
$$

where the explicit form of the coefficients $\mathscr{X}_n^{k,l}$ is given in appendix A. The blocks $\mathscr{Y}$ describe the scattering of states from V^{II}

$$
\mathscr{Y} : V^{\mathrm{II}} \longrightarrow V^{\mathrm{II}},
\tag{2.9}
$$

$$
|k,l\rangle_i^{\mathrm{II}} \;\mapsto\; \sum_{m=0}^{k+l} \sum_{j=1}^{4} \mathscr{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{II}}.
\tag{2.10}
$$

These S-matrix elements are given in eq. (5.18) of [35], but we will not need their general expression here. We will only need some special cases, which we have listed in appendix A. Finally, the middle block deals with the third case

$$
\mathscr{Z} : V^{\mathrm{III}} \longrightarrow V^{\mathrm{III}},
\tag{2.11}
$$

$$
|k,l\rangle_i^{\mathrm{III}} \;\mapsto\; \sum_{m=0}^{k+l} \sum_{j=1}^{6} \mathscr{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{III}},
\tag{2.12}
$$

with $\mathscr{Z}_{m;i}^{k,l;j}$ from eq. (6.11) in [35]. Similarly, their general expression is not needed for the sake of the present discussion, and we refer to [35] for details. Once again, the few ones that we will actually need are also listed in appendix A.

The S-matrix above is canonically normalized,[5] i.e., on the vector $|0,0\rangle_1^{\mathrm{III}} = w_1^{\ell_1} v_1^{\ell_2}$, the action of the S-matrix is $\mathbb{S}|0,0\rangle_1^{\mathrm{III}} = |0,0\rangle_1^{\mathrm{III}}$. The full $\mathrm{AdS}_5 \times \mathrm{S}^5$ string bound state S-matrix, in the $\mathfrak{su}(2)$ sector, is then obtained by taking two copies of the above S-matrix, and multiplying the result by a scalar factor. The latter is determined through the fusion procedure by using the scalar factor of the fundamental S-matrix [46, 47].

2.2 Monodromy and transfer matrices

Having introduced the relevant vector spaces and S-matrix, we are ready to define the corresponding monodromy and transfer matrix.

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[5] This agrees for example with the normalization of the matrix part of [35] and [17].

Consider K^{I} bound state particles with bound state numbers $\ell_1, \ldots, \ell_{K^{\mathrm{I}}}$ and momenta $p_1, \ldots, p_{K^{\mathrm{I}}}$. To these particles we add an auxiliary one, with momentum q and bound state number ℓ_0. Any state of this system lives in the following tensor product space

$$\mathcal{V} := V_{\ell_0}(q) \otimes V_{\ell_1}(p_1) \otimes \ldots \otimes V_{\ell_{K^{\mathrm{I}}}}(p_{K^{\mathrm{I}}}), \tag{2.13}$$

where V_{ℓ_i} is a carrier space of the bound state representation with the number ℓ_i. We split the states in this space into an auxiliary piece and a physical piece:

$$|A\rangle_0 \otimes |B\rangle_{K^{\mathrm{I}}} \in \mathcal{V},$$

where $|A\rangle_0 \in V_{\ell_0}(q)$ and[6] $|B\rangle_{K^{\mathrm{I}}} \in V_P := \bigotimes_i V_{\ell_i}(p_i)$. The monodromy matrix acting in the space $\mathcal{V}$ is defined as follows

$$\mathcal{T}_{\ell_0}(q|\vec{p}) := \prod_{i=1}^{K^{\mathrm{I}}} \mathbb{S}_{0k}(q, p_i), \tag{2.14}$$

where $\mathbb{S}_{0k}(q, p_k)$ is the bound state S-matrix describing scattering between the auxiliary particle and a 'physical' particle with momentum p_k and bound state number ℓ_k.

The monodromy matrix can be seen as a $4\ell_0 \times 4\ell_0$ dimensional matrix in the auxiliary space $V_{\ell_0}(q)$, the corresponding matrix elements being themselves operators on V_P. Indeed, introducing a basis $|e_I\rangle$ for $V_{\ell_0}(q)$, with the index I labelling a $4\ell_0$-dimensional space, and a basis $|f_A\rangle$ for V_P, the action of the monodromy matrix $\mathcal{T} \equiv \mathcal{T}_{\ell_0}(q|\vec{p})$ on the total space $\mathcal{V}$ can be written as

$$\mathcal{T}(|e_I\rangle \otimes |f_A\rangle) = \sum_{J,B} T_{IA}^{JB} |e_J\rangle \otimes |f_B\rangle. \tag{2.15}$$

The matrix entries of the monodromy matrix can then be denoted as

$$\mathcal{T}|e_I\rangle = \sum_J \mathcal{T}_I^J |e_J\rangle, \tag{2.16}$$

while the action of the matrix elements $\mathcal{T}_I^J$ as operators on V_P can easily be read off:

$$\mathcal{T}_I^J |f_A\rangle = \sum_B T_{IA}^{JB} |f_B\rangle. \tag{2.17}$$

The operators $\mathcal{T}_I^J$ have non-trivial commutation relations among themselves. Consider two different auxiliary spaces $V_{\ell_0}(q), V_{\tilde{\ell}_0}(\tilde{q})$. The Yang-Baxter equation for $\mathbb{S}$ implies that

$$\mathbb{S}(q, \tilde{q}) \mathcal{T}_{\ell_0}(q|\vec{p}) \mathcal{T}_{\tilde{\ell}_0}(\tilde{q}|\vec{p}) = \mathcal{T}_{\tilde{\ell}_0}(\tilde{q}|\vec{p}) \mathcal{T}_{\ell_0}(q|\vec{p}) \mathbb{S}(q, \tilde{q}), \tag{2.18}$$

where $\mathbb{S}(q, \tilde{q})$ is the S-matrix describing the scattering between two bound state particles of bound state numbers $\ell_0, \tilde{\ell}_0$ and momenta $q, \tilde{q}$ respectively. By explicitly working out these relations, one finds the commutation relations between the different matrix elements of the monodromy matrix. The fundamental commutation relations (2.18) constitute a cornerstone of the Algebraic Bethe Ansatz [48].

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[6]All the tensor products are defined with increasing order of the index as $1, 2, \ldots, K$.

It is convenient to pick up the following explicit basis $|e_I\rangle$ in the space $V_{\ell_0}(q)$

$$
\begin{aligned}
e_{\alpha;k} &:= \theta_\alpha w_1^{\ell_0-k-1} w_2^k, \\
e_k &:= w_1^{\ell_0-k} w_2^k, \\
e_{34;k} &:= \theta_3 \theta_4 w_1^{\ell_0-k-1} w_2^{k-1}.
\end{aligned}
\tag{2.19}
$$

The transfer matrix is then defined as

$$
\mathscr{T}_0(q|\vec{p}) := \mathrm{str}_0 \mathcal{T}_{\ell_0}(q|\vec{p}),
\tag{2.20}
$$

and it can be viewed as an operator acting on the physical space V_P. In terms of the operator entries of the monodromy matrix, the transfer matrix is written as

$$
\mathscr{T}_0(q|\vec{p}) = \sum_{k=0}^{\ell_0} \mathcal{T}_k^k + \sum_{k=1}^{\ell_0-1} \mathcal{T}_{34;k}^{34;k} - \sum_{k=0}^{\ell_0-1} \sum_{\alpha=3,4} \mathcal{T}_{\alpha;k}^{\alpha;k}.
\tag{2.21}
$$

In the remainder of this paper we will study the action of $\mathscr{T}_0(q|\vec{p})$ on the physical space in detail and derive its eigenvalues.

3 Diagonalization of the transfer matrix

An efficient way to find the eigenvalues of the transfer matrix is to use the Algebraic Bethe Ansatz. We start by defining a vacuum state

$$
|0\rangle_P = w_1^{\ell_1} \otimes \ldots \otimes w_1^{\ell_{K^{\mathrm{I}}}}.
\tag{3.1}
$$

We then compute the action of the transfer matrix on this state, which appears to be one of its eigenstates, and afterwards use specific elements of the monodromy matrix to generate the whole spectrum of eigenvalues. Imposing the eigenstate condition should result in the determination of the full set of eigenvalues and associated Bethe equations, therefore providing the complete solution of the asymptotic spectral problem.

3.1 Eigenvalue of the transfer matrix on the vacuum

As promised, we first deduce the action of the transfer matrix on the vacuum. We will do this for each of the separate sums in (2.21). Let us start with the fermionic part, i.e. we want to compute

$$
\sum_{k=0}^{\ell_0-1} \mathcal{T}_{\alpha;k}^{\alpha;k} |0\rangle_P, \qquad \alpha = 3, 4.
\tag{3.2}
$$

Taking into account the explicit form of the S-matrix elements entering the monodromy matrix, we find that the only contribution to $\mathcal{T}_{\alpha;k}^{\alpha;k}|0\rangle$ comes from diagonal scattering elements. To be precise, one finds

$$
\mathcal{T}_{\alpha;k}^{\alpha;k}|0\rangle_P = \prod_i \mathscr{Y}_{k;1}^{k,0;1}(q, p_i)|0\rangle_P,
\tag{3.3}
$$

where $\mathscr{Y}_{k;1}^{k,0;1}(q, p_i)$ are Case II S-matrix elements. By explicitly working out this expression, one finds

$$\mathscr{Y}_{k;1}^{k,0;1}(q, p_i) = \frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \mathscr{X}_k^{k,0}(q, p_i), \qquad (3.4)$$

where $x_0^\pm$ are defined in terms of the momentum q, and one uses equation (4.13) of [35]:

$$\mathscr{X}_k^{k,0}(q, p_i) = \mathcal{D} \frac{\prod_{j=0}^{k-1} u_0 - u_i + \frac{\ell_0 - \ell_i - 2j}{2}}{\prod_{j=1}^{k} u_0 - u_i + \frac{\ell_0 + \ell_i - 2j}{2}} \qquad k = 1, \cdots, \ell_0 - 1,$$

$$\mathscr{X}_0^{0,0}(q, p_i) = \mathcal{D} \frac{\pi}{\sin(\pi \ell_i)\Gamma(1 - \ell_i)\Gamma(\ell_i)} = \mathcal{D} = \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-} \frac{x_i^-}{x_i^+}}. \qquad (3.5)$$

Obviously, the contribution of $\mathcal{T}_{\alpha;k}^{\alpha;k}$ is the same for $\alpha = 3, 4$. Here x_m, $m = 0, 1, \ldots K^I$, are the constrained parameters (λ is the 't Hooft coupling)

$$x_m^+ + \frac{1}{x_m^+} - x_m^- - \frac{1}{x_m^-} = 2\ell_m \frac{i}{g}, \qquad g = \frac{\sqrt{\lambda}}{2\pi}$$

related to the particle momenta as $p_m = \frac{1}{i} \log\left(\frac{x_m^+}{x_m^-}\right)$. Also, u_m represents the corresponding rapidity variable given by

$$x_m^\pm + \frac{1}{x_m^\pm} = \frac{2i}{g} u_m \pm \frac{i}{g} \ell_m. \qquad (3.6)$$

Next, we consider the more involved bosonic part. This can be written as

$$\mathcal{T}_0^0 + \mathcal{T}_{\ell_0}^{\ell_0} + \sum_{k=1}^{\ell_0 - 1} \left\{ \mathcal{T}_k^k + \mathcal{T}_{34;k}^{34;k} \right\}. \qquad (3.7)$$

We first determine $\mathcal{T}_0^0 |0\rangle_P$ and $\mathcal{T}_{\ell_0}^{\ell_0} |0\rangle_P$. For these operators, one again finds that only diagonal scattering elements of the S-matrices contribute, which leads to

$$\mathcal{T}_0^0 |0\rangle_P = \prod_i \mathscr{X}_{0;1}^{0,0;1}(q, p_i) |0\rangle_P,$$

$$\mathcal{T}_{\ell_0}^{\ell_0} |0\rangle_P = \prod_i \mathscr{X}_{\ell_0;1}^{\ell_0,0;1}(q, p_i) |0\rangle_P. \qquad (3.8)$$

These matrix elements can be computed explicitly and give

$$\mathcal{T}_0^0 |0\rangle_P = |0\rangle_P, \qquad (3.9)$$

$$\mathcal{T}_{\ell_0}^{\ell_0} |0\rangle_P = \left\{ \prod_{i=1}^{K^I} \frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathscr{X}_{\ell_0}^{\ell_0,0}(q, p_i) \right\} |0\rangle_P. \qquad (3.10)$$

where we define

$$\mathscr{X}_{\ell_0}^{\ell_0,0}(q, p_i) = \mathcal{D} \frac{\prod_{j=0}^{\ell_0 - 1} u_0 - u_i + \frac{\ell_0 - \ell_i - 2j}{2}}{\prod_{j=1}^{\ell_0} u_0 - u_i + \frac{\ell_0 + \ell_i - 2j}{2}}. \qquad (3.11)$$

The next thing to consider is the sum

$$\sum_{k=1}^{\ell_0-1} \left\{ \mathcal{T}_k^k + \mathcal{T}_{34;k}^{34;k} \right\}. \tag{3.12}$$

While in the previous computations one could simply restrict to the diagonal elements, one obtains instead a matrix structure for this last piece. This is due to the fact that there are scattering processes that relate $w_2 \leftrightarrow \theta_\alpha \theta_\beta$. To be more precise, for the action of $\mathcal{T}_k^k$ and $\mathcal{T}_{34,k}^{34,k}$ one finds

$$\mathcal{T}_k^k|0\rangle_P = \sum_{a_1\ldots a_{K^{\mathrm{I}}}=1,3} \mathscr{L}_{k;1}^{k,0;a_1}(q,p_1)\, \mathscr{L}_{k;a_1}^{k,0;a_2}(q,p_2)\ldots \mathscr{L}_{k;a_{K^{\mathrm{I}}}}^{k,0;1}(q,p_{K^{\mathrm{I}}})|0\rangle_P, \tag{3.13}$$

$$\mathcal{T}_{34,k}^{34,k}|0\rangle_P = \sum_{a_1\ldots a_{K^{\mathrm{I}}}=1,3} \mathscr{L}_{k;3}^{k,0;a_1}(q,p_1)\, \mathscr{L}_{k;a_1}^{k,0;a_2}(q,p_2)\ldots \mathscr{L}_{k;a_{K^{\mathrm{I}}}}^{k,0;3}(q,p_{K^{\mathrm{I}}})|0\rangle_P. \tag{3.14}$$

In order to evaluate the above expressions explicitly, it proves useful to use a slightly more general reformulation.[7] One can reintroduce the elements $\mathcal{T}_k^{34,k}$ and $\mathcal{T}_{34,k}^k$ from the monodromy matrix. Their action on the vacuum is

$$\mathcal{T}_k^{34,k}|0\rangle_P = \sum_{a_1\ldots a_{K^{\mathrm{I}}}=1,3} \mathscr{L}_{k;1}^{k,0;a_1}(q,p_1)\, \mathscr{L}_{k;a_1}^{k,0;a_2}(q,p_2)\ldots \mathscr{L}_{k;a_{K^{\mathrm{I}}}}^{k,0;3}(q,p_{K^{\mathrm{I}}})|0\rangle_P, \tag{3.15}$$

$$\mathcal{T}_{34,k}^k|0\rangle_P = \sum_{a_1\ldots a_{K^{\mathrm{I}}}=1,3} \mathscr{L}_{k;3}^{k,0;a_1}(q,p_1)\, \mathscr{L}_{k;a_1}^{k,0;a_2}(q,p_2)\ldots \mathscr{L}_{k;a_{K^{\mathrm{I}}}}^{k,0;1}(q,p_{K^{\mathrm{I}}})|0\rangle_P. \tag{3.16}$$

They describe the mixing between the states $|e_{34,k}\rangle$ and $|e_k\rangle$. If we consider the two-dimensional vector space spanned by $|e_{34,k}\rangle$ and $|e_k\rangle$ for fixed $k \in \{1,\ldots,\ell_0-1\}$, we see that the above elements define a 2×2 dimensional matrix

$$\mathcal{T}_{2\times 2} = \begin{pmatrix} \mathcal{T}_k^k & \mathcal{T}_k^{34,k} \\ \mathcal{T}_{34,k}^k & \mathcal{T}_{34,k}^{34,k} \end{pmatrix}, \tag{3.17}$$

and the bosonic part of the transfer matrix is just the trace of this matrix. Moreover, it is easily seen from the definition of the transfer matrix that this matrix factorizes

$$\mathcal{T}_{2\times 2} = \prod_{i=1}^{K} \begin{pmatrix} \mathscr{L}_{k;1}^{k,0;1}(q,p_i) & \mathscr{L}_{k;1}^{k,0;3}(q,p_i) \\ \mathscr{L}_{k;3}^{k,0;1}(q,p_i) & \mathscr{L}_{k;3}^{k,0;3}(q,p_i) \end{pmatrix}. \tag{3.18}$$

The trace of this matrix is given by the sum of its eigenvalues, hence it remains to find the eigenvalues of this matrix. Actually, it is easily checked that the eigenvectors of

$$\begin{pmatrix} \mathscr{L}_{k;1}^{k,0;1}(q,p_i) & \mathscr{L}_{k;1}^{k,0;3}(q,p_i) \\ \mathscr{L}_{k;3}^{k,0;1}(q,p_i) & \mathscr{L}_{k;3}^{k,0;3}(q,p_i) \end{pmatrix} \tag{3.19}$$

are independent of p_i. In other words, these are automatically eigenvectors of $\mathcal{T}_{2\times 2}$, and the corresponding eigenvalues are the product of the eigenvalues of the above matrices.

---

[7]We remark that this computation has been performed at weak coupling in [17, 19].

The individual eigenvalues are given by

$$\lambda_\pm(q, p_i, k) = \frac{\mathscr{X}_k^{k,0}}{2\mathcal{D}}\left[1 - \frac{(x_i^- x_0^+ - 1)(x_0^+ - x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)} + \frac{2ik}{g}\frac{x_0^+(x_i^- + x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)}\right.$$

$$\left.\pm\frac{ix_0^+(x_i^- - x_i^+)}{(x_i^- - x_0^+)(x_0^+ x_i^+ - 1)}\sqrt{\left(\frac{2k}{g}\right)^2 + 2i\left[x_0^+ + \frac{1}{x_0^+}\right]\frac{2k}{g} - \left[x_0^+ - \frac{1}{x_0^+}\right]^2}\right]. \tag{3.20}$$

The action of the transfer matrix on the vacuum is now given by the sum of all the above terms. From this it is easily seen that the vacuum is indeed an eigenvector of the transfer matrix with the following eigenvalue

$$\Lambda(q|\vec{p}) = 1 + \prod_{i=1}^{K^I}\left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)}\sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}}\mathscr{X}_{\ell_0}^{\ell_0,0}\right]$$

$$-2\sum_{k=0}^{\ell_0-1}\prod_{i=1}^{K^I}\left(\frac{x_0^+ - x_i^+}{x_0^- - x_i^+}\sqrt{\frac{x_0^-}{x_0^+}}\left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}}\right]\mathscr{X}_k^{k,0}\right)$$

$$+\sum_{k=1}^{\ell_0-1}\prod_{i=1}^{K^I}\lambda_+(q, p_i, k) + \sum_{k=1}^{\ell_0-1}\prod_{i=1}^{K^I}\lambda_-(q, p_i, k). \tag{3.21}$$

For the fundamental case ($\ell_0 = \ell_i = 1 \ \forall i$), this reduces to

$$\mathcal{T}_0(q|\vec{p})|0\rangle_P = \left\{\prod_i \mathscr{X}_{0;1}^{0,0;1}(q, p_i) + \prod_i \mathscr{X}_{1;1}^{1,0;1}(q, p_i) - 2\prod_i \mathscr{Y}_{0;1}^{0,0;1}(q, p_i)\right\}|0\rangle_P$$

$$= \left\{1 + \prod_{i=1}^{K^I}\frac{1 - \frac{1}{x_0^- x_i^+}}{1 - \frac{1}{x_0^- x_i^-}}\frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} - 2\prod_{i=1}^{K^I}\frac{x_0^+ - x_i^+}{x_0^+ - x_i^-}\sqrt{\frac{x_i^-}{x_i^+}}\right\}|0\rangle_P. \tag{3.22}$$

We would like to point out that the square roots in the eigenvalues $\lambda_\pm$ will never appear in the vacuum eigenvalue. This is because the square root part only depends on the auxiliary momentum q, and it can be seen that, after summing the contribution from λ_+ and λ_-, only even powers of this square root piece survive.

3.2 Creation operators and excited states

The next step in the algebraic Bethe ansatz is to introduce creation operators. These operators will be entries from our monodromy matrix. By acting with those operators on the vacuum one creates new (excited) states, which again will be eigenstates of the transfer matrix. We will need to specify which monodromy matrix entries correspond to creation operators for our purposes.

Recall that, from the symmetry invariance of the S-matrix, one can deduce that the quantum numbers K^{II} (total number of fermions) and K^{III} (total number of fermions of a definite species, say, 3) are conserved. Any element $\mathcal{T}_I^J$ is called a creation operator if $K^{II}(|e_I\rangle_0) > K^{II}(|e_J\rangle_0)$, it is called an annihilation operator if $K^{II}(|e_I\rangle_0) < K^{II}(|e_J\rangle_0)$ and diagonal if $K^{II}(|e_I\rangle_0) = K^{II}(|e_J\rangle_0)$. The reason for this assignment is the following.

Consider a creation operator $\mathcal{T}_I^J$ and any physical state $|A\rangle_P$. The action of a creation operator is defined via (2.15). Since the total number K^{II} is preserved, and the K^{II} charge in the auxiliary space has decreased, it has necessarily increased in the physical space. The number K^{II} corresponds to the number of fermions in the system, hence, by acting with $\mathcal{T}_I^J$ on $|A\rangle_P$, one creates extra fermions in the physical space. Notice that this also implies that acting with an annihilation operator on the vacuum annihilates it, whence the name.

We will create excited states by considering fundamental auxiliary spaces with momenta λ_i. Since these are fundamental spaces, their monodromy matrices are only 4×4-dimensional. Our discussion will be very similar to the treatment of the algebraic Bethe ansatz for the Hubbard model which was first performed in [49, 50]. In order to make contact with the treatment of [36] and with the standard notation used for the Hubbard model, we parameterize this monodromy matrix as

$$\begin{pmatrix} B & B_3 & B_4 & F \\ C_3 & A_3^3 & A_4^3 & B_3^* \\ C_4 & A_3^4 & A_4^4 & B_4^* \\ C & C_3^* & C_4^* & D \end{pmatrix}. \tag{3.23}$$

Notice that one finds two seemingly different sets of creation operators $B_3(\lambda_i), B_4(\lambda_i), F(\lambda_i)$ and $B_3^*(\lambda_i), B_4^*(\lambda_i), F(\lambda_i)$. As discussed in [49], it is enough to restrict to one set. In what follows, we will use the operators $B_3(\lambda_i), B_4(\lambda_i), F(\lambda_i)$ to create fermionic excitations out of the vacuum.

A generic excited state will now be formed by acting with a number of those operators on the vacuum, e.g. one can consider states like

$$B_3(\lambda_1)B_4(\lambda_2)|0\rangle. \tag{3.24}$$

To find out whether this is an eigenstate of the transfer matrix, one has to commute the diagonal elements of the transfer matrices through the creation operators and let them act on the vacuum. Imposing the eigenstate condition will in general give constraints on the momenta λ_i. The explicit commutation relations will be the subject of the next section.

3.3 Commutation relations

In order to compute the action of the transfer matrix on an excited state, we need to compute the commutation relations between the diagonal elements $\mathcal{T}_A^A$ and the aforementioned creation operators. While we have to use creation operators in a fundamental auxiliary representation, the diagonal elements are to be taken in the bound state representation with generic ℓ_0. The commutation relations follow from (2.18). We will report the complete derivation of one specific commutation relation, and only give the final result for the remaining ones. In the derivation, one has to pay particular attention to the fermionic nature of the operators.

Consider the operator $B_3(\lambda)$ and the element $\mathcal{T}_{3,k}^{3,k}$ from the transfer matrix. From (2.18), one finds

$$\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)\mathcal{T}(q)\mathcal{T}(\lambda)e_{3,k}\tilde{e}_{3,0} = \mathbb{P}_{3,k|0}\mathcal{T}(\lambda)\mathcal{T}(q)\mathbb{S}(q,\lambda)e_{3,k}\tilde{e}_{3,0}, \tag{3.25}$$

where we have dropped the indices ℓ_0 and $\tilde{\ell}_0 = 1$, and where the tilde on $\tilde{e}_{3,0}$ denotes a basis element in the second auxiliary space. The operator $\mathbb{P}_{A|B}$ is the projection operator onto the subspace generated by the basis element $e_A \tilde{e}_B$. The right hand side of the above equation gives

$$
\begin{aligned}
\mathbb{P}_{3,k|0} \mathcal{T}(\lambda)\mathcal{T}(q)\mathbb{S}(q,\lambda)e_{3,k}\tilde{e}_{3,0} &= \mathbb{P}_{3,k|0}\mathscr{X}_k^{k,0}\mathcal{T}(\lambda)\mathcal{T}(q)e_{3,k}\tilde{e}_{3,0} \\
&= \mathbb{P}_{3,k|0}\sum_{A,B}\mathscr{X}_k^{k,0}(-1)^{F_A}(\mathcal{T}_3^B\tilde{e}_B)(\lambda)(\mathcal{T}_{3,k}^A(q)e_A) \\
&= \mathscr{X}_k^{k,0}(-1)^{F(3,k)}(\mathcal{T}_3^0\tilde{e}_0)(\lambda)(\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}) \\
&= -\mathscr{X}_k^{k,0}B_3^0(\lambda)\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\tilde{e}_0.
\end{aligned} \tag{3.26}
$$

The left hand side reduces to

$$
\begin{aligned}
\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)\mathcal{T}(q)\mathcal{T}(\lambda)e_{3,k}\tilde{e}_{3,0} &= -\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)\mathcal{T}(q)(\mathcal{T}(\lambda)_3^B\tilde{e}_B)e_{3,k} \tag{3.27} \\
&= -\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)\mathcal{T}(q)\left\{\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_3^3(\lambda)\tilde{e}_3 + \mathcal{T}_3^1(\lambda)\tilde{e}_1\right\}e_{3,k} \\
&= \mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)(\mathcal{T}_{3,k}^A(q)e_A)\left\{\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_3^3(\lambda)\tilde{e}_3 + \mathcal{T}_3^1(\lambda)\tilde{e}_1\right\}.
\end{aligned}
$$

Because of the projection, we only need to take into account terms that are mapped onto $e_{3,k}\tilde{e}_0$ by the action of the S-matrix. These are given by

$$
\begin{aligned}
\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)&\left\{\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_{3,k}^{3,k-1}(q)e_{3,k-1}\mathcal{T}_3^1(\lambda)\tilde{e}_1 + \mathcal{T}_{3,k}^k(q)e_k\mathcal{T}_3^3(\lambda)\tilde{e}_{3,0}\right\} = \\
&\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)\left\{-\mathcal{T}_{3,k}^{3,k}(q)\mathcal{T}_3^0(\lambda)e_{3,k}\tilde{e}_0 - \mathcal{T}_{3,k}^{3,k-1}(q)\mathcal{T}_3^1(\lambda)e_{3,k-1}\tilde{e}_1 + \right. \\
&\left. + \mathcal{T}_{3,k}^k(q)\mathcal{T}_3^3(\lambda)e_k\tilde{e}_{3,0} + \mathcal{T}_{3,k}^{34,k-1}(q)\mathcal{T}_3^3(\lambda)e_{34,k-1}\tilde{e}_{3,0}\right\}.
\end{aligned} \tag{3.28}
$$

Working this out explicitly yields

$$
\begin{aligned}
\mathbb{P}_{3,k|0}\mathbb{S}(q,\lambda)&\left\{\mathcal{T}_{3,k}^{3,k}(q)e_{3,k}\mathcal{T}_3^0(\lambda)\tilde{e}_0 + \mathcal{T}_{3,k}^{3,k-1}(q)e_{3,k-1}\mathcal{T}_3^1(\lambda)\tilde{e}_1 + \mathcal{T}_{3,k}^k(q)e_k\mathcal{T}_3^3(\lambda)\tilde{e}_{3,0}\right\} = \\
&\left\{-\mathcal{T}_{3,k}^{3,k}(q)\mathcal{T}_3^0(\lambda)\mathscr{Y}_{k;1}^{k,0;1} - \mathscr{Y}_{k;1}^{k-1,1;1}\mathcal{T}_{3,k}^{3,k-1}(q)\mathcal{T}_3^1(\lambda) + \right. \\
&\left. + \mathscr{Y}_{k;2}^{k,1;1}\mathcal{T}_{3,k}^k(q)\mathcal{T}_3^3(\lambda) + \mathscr{Y}_{k;4}^{k,1;1}\mathcal{T}_{3,k}^{34,k-1}(q)\mathcal{T}_3^3(\lambda)\right\}e_{3,k}\tilde{e}_0.
\end{aligned} \tag{3.29}
$$

From this we now read off the final commutation relation[8]

$$
\begin{aligned}
\mathscr{X}_k^{k,0}B_3(\lambda)\mathcal{T}_{3,k}^{3,k}(q) &= \mathscr{Y}_{k;1}^{k,0;1}\mathcal{T}_{3,k}^{3,k}(q)B_3(\lambda) + \mathscr{Y}_{k;1}^{k-1,1;1}\mathcal{T}_{3,k}^{3,k-1}(q)C_3^*(\lambda) + \\
&\quad -\mathscr{Y}_{k;2}^{k,1;1}\mathcal{T}_{3,k}^k(q)A_3^3(\lambda) - \mathscr{Y}_{k;4}^{k,1;1}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda).
\end{aligned} \tag{3.30}
$$

Notice that in the above relation the operators are ordered in such a way that all annihilation and diagonal elements are on the right. This is done because the action of those elements on the vacuum is known. We would also like to compare these commutation relations with [49, 50] for the Hubbard model. We see that the first and third term are also present in the Hubbard model. However, due to the fact that we are dealing with bound state representation, we also obtain *two additional terms*.

---

[8]Throughout the rest of this section 3.3, if not otherwise indicated, the coefficient functions appearing have to be understood as $\mathscr{X} \equiv \mathscr{X}(q,\lambda)$, $\mathscr{Y} \equiv \mathscr{Y}(q,\lambda)$, $\mathscr{Z} \equiv \mathscr{Z}(q,\lambda)$ (indices are omitted here for simplicity).

Generically, the commutation relations produce "wanted" terms, which are those which directly contribute to the eigenvalue, and other "unwanted" terms. The latter terms are those which need to vanish, in order for the state of our ansatz to be an eigenstate. In (3.30), one can easily see by acting on the vacuum that the wanted term is the first term on the right hand side, while the other terms are unwanted. The cancellation of the unwanted terms will give rise to certain constraints, which are precisely the auxiliary Bethe equations.

The other commutation relations one needs to compute are those with $\mathcal{T}_k^k, \mathcal{T}_{34,k}^k$ and $\mathcal{T}_{4,k}^{4,k}$. Their derivation is considerably more involved, especially the procedure of reordering them according to the above "annihilation and diagonal on the right" prescription. We will present the commutation relations we will actually need in the coming sections. We will give the wanted terms, and focus on one specific type of unwanted terms. Schematically, we will focus on the following structure:

$$\left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q)\right] B_\alpha(\lambda) = \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} B_\alpha(\lambda) \left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q)\right] + \frac{\mathscr{Y}_{k;2}^{k,0;1}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{T}_{\alpha,k}^k(q) B(\lambda) + \ldots$$

$$\mathcal{T}_{\alpha_1,k}^{\gamma,k}(q) B_{\alpha_2}(\lambda) = \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} B_{\beta_2}(\lambda) \mathcal{T}_{\beta_1,k}^{\gamma,k}(q) r_{\alpha_1\alpha_2}^{\beta_1\beta_2} \left(u_0 + \frac{\ell_0 - 1}{2} - k, u_\lambda\right) + \quad (3.31)$$

$$+ \frac{\mathscr{Y}_{k;2}^{k,0;1}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{T}_{\beta_1,k}^k(q) A_{\beta_2}^\gamma(\lambda) r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_\lambda, u_\lambda) + \ldots$$

Here, u_λ is given by (cf. (3.6))

$$u_\lambda = \frac{g}{2i}\left(x^+(\lambda) + \frac{1}{x^+(\lambda)} - \frac{i}{g}\right)$$

and $r_{\alpha\beta}^{\gamma\delta}(u_\lambda, u_\mu)$ are the components of the 6-vertex model S-matrix (B.3) with $U = -1$. We would like to point out that, when comparing this structure against formulas (34-36) of [49], one immediately recognizes a similarity between the commutation relations. As was shown in [36], for the case in which all representations are taken to be fundamental, the commutation relations do agree. The additional contributions coming from the fact that we are dealing with bound states in e.g. (3.30), will only generate a new class of unwanted terms. Hence, these new terms will not contribute to the eigenvalues.

Let us mention one commutation relation which is particularly straightforward to derive, namely, the one between two fermionic creation operators, as found from (2.18) with $\ell_0 = \tilde{\ell}_0 = 1$. This relation reads

$$B_\alpha(\lambda) B_\beta(\mu) = -\mathscr{X}_0^{0,0}(\lambda, \mu) B_\delta(\mu) B_\gamma(\lambda) r_{\alpha\beta}^{\gamma\delta}(u_\lambda, u_\mu)$$

$$+ \frac{\mathscr{L}_{1;6}^{1,0;1}(\lambda, \mu)}{\mathscr{L}_{1;1}^{1,0;1}(\lambda, \mu)} \left[F(\lambda) B(\mu) - F(\mu) B(\lambda)\right] \epsilon_{\alpha\beta}. \quad (3.32)$$

This reproduces the result of [36], and, in this way, one can see the emergence of nesting. As a matter of fact, in [36, 49, 50] the appearance of the 6-vertex model S-matrix was used to completely fix the form of the excited eigenstates, and this can also be done in our case.

3.4 First excited state

The first excited state is of the form

$$|1\rangle = \mathcal{F}^\alpha B_\alpha(\lambda)|0\rangle_P, \tag{3.33}$$

where we sum over the repeated fermionic index. This state has $K^{\mathrm{II}} = 1$. As previously discussed, all the commutation relations are ordered in such a way that all annihilation and diagonal operators are on the right. From the commutation relations (3.31) one finds

$$
\left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] \mathcal{F}^\alpha B_\alpha(\lambda)|0\rangle_P = \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{F}^\alpha B_\alpha(\lambda) \left[\mathcal{T}_k^k(q) + \mathcal{T}_{\alpha\beta,k}^{\alpha\beta,k}(q) \right] |0\rangle_P
$$
$$
+ \frac{\mathscr{Y}_{k;2}^{k,0;1}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{F}^\alpha \mathcal{T}_{\alpha,k}^k(q) B(\lambda)|0\rangle_P, \tag{3.34}
$$

$$
\left[\mathcal{T}_{\alpha_1,k}^{\alpha_1,k}(q) \right] \mathcal{F}^{\alpha_2} B_{\alpha_2}(\lambda)|0\rangle_P = \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{F}^{\alpha_2} r_{\alpha_1\alpha_2}^{\beta_1\beta_2} \left(u_0 + \frac{\ell_0 - 1}{2} - k, u_\lambda \right) B_{\beta_2}(\lambda) \left[\mathcal{T}_{\alpha_1,k}^{\beta_1,k}(q) \right] |0\rangle_P
$$
$$
+ \frac{\mathscr{Y}_{k;2}^{k,0;1}}{\mathscr{Y}_{k;1}^{k,0;1}} \mathcal{F}^{\alpha_2} r_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u_\lambda, u_\lambda) \mathcal{T}_{\beta_2,k}^k(q) A_{\beta_1}^{\alpha_1}(\lambda)|0\rangle_P, \tag{3.35}
$$

where we remind that we concentrate on only one type of unwanted terms, for the sake of clarity. The coefficient functions appearing in the above two formulas have to be understood as $\mathscr{X} \equiv \mathscr{X}(q,\lambda)$, $\mathscr{Y} = \mathscr{Y}(q,\lambda)$ (indices are omitted here for simplicity).

Since $\mathcal{T}_{\beta,k}^{\alpha,k}|0\rangle_P \sim \delta_\beta^\alpha|0\rangle_P$, we find that $|1\rangle$ can only be an eigenstate of the transfer matrix if

$$\mathcal{F}^\alpha r_{\gamma\alpha}^{\gamma\beta} \left(u_0 + \frac{\ell_0 - 1}{2} - k, u_\lambda \right) \sim \mathcal{F}^\beta. \tag{3.36}$$

This means that $\mathcal{F}^\alpha$ is an eigenvector of the transfer matrix of the 6-vertex model. Luckily, one finds that the eigenstates of the 6-vertex model are independent of the auxiliary momenta. This means that the k dependence in the above r-matrix only appears in the eigenvalue $\Lambda^{(6v)}$, where $\Lambda^{(6v)}$ is the eigenvalue of the auxiliary 6-vertex model. From (B.15) we find ($K = K^{\mathrm{II}} = 1$)

$$
\Lambda^{(6v)}(u_0|u_\lambda) = \prod_{i=1}^{K^{\mathrm{III}}} \frac{1}{b\left(w_i, u_0 + \frac{\ell_0-1}{2} - k \right)} + b\left(u_0 + \frac{\ell_0-1}{2} - k, u_\lambda \right) \prod_{i=1}^{K^{\mathrm{III}}} \frac{1}{b\left(u_0 + \frac{\ell_0-1}{2} - k, w_i \right)},
$$

together with the auxiliary equation (B.16)

$$b(w_j, u_\lambda) = \prod_{i=1, i\neq j}^{K^{\mathrm{III}}} \frac{b(w_j, w_i)}{b(w_i, w_j)}. \tag{3.37}$$

We also have to deal with the unwanted terms. Here we remark that, since we have chosen $\mathcal{F}^\alpha$ to be an eigenvector of the 6-vertex S-matrix, this also affects the unwanted terms. One explicitly finds that they are proportional to

$$\left\{ \Lambda^{(6v)}(u_\lambda|u_\lambda) A_\alpha^\alpha(\lambda) - B(\lambda) \right\} |0\rangle_P. \tag{3.38}$$

Explicitly working this out this leads us to the following auxiliary Bethe equations:

$$\prod_{i=1}^{K^{\mathrm{I}}} \frac{x^+(\lambda) - x_i^-}{x^+(\lambda) - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} = \Lambda^{(6v)}(u_\lambda | u_\lambda). \tag{3.39}$$

In order to make contact with the bound state Bethe equations found in [37], let us define $y \equiv x^+(\lambda)$ and rescale $w \to \frac{g}{2i} w$. We find that $|1\rangle$ is an eigenstate, provided the auxiliary Bethe equations hold:[9]

$$\prod_{i=1}^{K^{\mathrm{I}}} \frac{y - x_i^-}{y - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} = \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - y - \frac{1}{y} - \frac{i}{g}}{w_i - y - \frac{1}{y} + \frac{i}{g}},$$

$$\frac{w_i - y - \frac{1}{y} + \frac{i}{g}}{w_i - y - \frac{1}{y} - \frac{i}{g}} = \prod_{j=1, j\neq i}^{K^{\mathrm{III}}} \frac{w_i - w_j + \frac{2i}{g}}{w_i - w_j - \frac{2i}{g}}. \tag{3.40}$$

This exactly matches with the auxiliary bound state Bethe ansatz equations. The corresponding eigenvalue is

$$\Lambda(q|\vec{p}) = \frac{y - x_0^-}{y - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \tag{3.41}$$

$$+ \frac{y - x_0^-}{y - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2i\ell_0}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^- x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathscr{X}_{\ell_0}^{\ell_0, 0} \right]$$

$$+ \sum_{k=1}^{\ell_0 - 1} \frac{y - x_0^-}{y - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2ik}{g}} \right] \left\{ \prod_{i=1}^{K^{\mathrm{I}}} \lambda_+(q, p_i, k) + \prod_{i=1}^{K^{\mathrm{I}}} \lambda_-(q, p_i, k) \right\}$$

$$- \sum_{k=0}^{\ell_0 - 1} \frac{y - x_0^-}{y - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y}}{x_0^+ + \frac{1}{x_0^+} - y - \frac{1}{y} - \frac{2ik}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \times$$

$$\times \mathscr{X}_k^{k, 0} \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k-1)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} + \frac{y + \frac{1}{y} - x_0^+ - \frac{1}{x_0^+} + \frac{2ik}{g}}{y + \frac{1}{y} - x_0^+ - \frac{1}{x_0^+} + \frac{2i(k+1)}{g}} \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+3)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} \right\}.$$

We stress once again that the above eigenvalue is for the canonically normalized S-matrix. The dependence of Λ on the bound state numbers of the physical particles is hidden in their parameters $x_i^\pm$ and in the S-matrix element $\mathscr{X}$. Notice that, when projected in the fundamental representation, the formula above reproduces the result of [36].

3.5 General result and Bethe equations

As was stressed before, by comparing our commutation relations against (34)-(36) from [49, 50], one immediately notices several similarities. It turns out that one can closely follow the derivation presented in those papers, and from the diagonal terms read off the general eigenvalue. Furthermore, cancelling the first few unwanted terms reveals itself as sufficient to derive the complete set of auxiliary Bethe equations.

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[9]We remark that, for $K^{\mathrm{III}} = 0$, the solution of (3.40) correspond to the highest weight state of the auxiliary six-vertex model, while, for $K^{\mathrm{III}} = 1$, one formally obtains a solution only if some of the auxiliary roots are equal to infinity. This corresponds to a descendent of the highest weight state under the $\mathfrak{su}(2)$ symmetry.

More specifically, the results of appendix C and the previously known results for the case when all physical legs are in the fundamental representation indicate the generalization of the formula for the transfer-matrix eigenvalues to multiple excitations. In terms of S-matrix elements, this generalization is given by

$$\Lambda(q|\vec{p}) = \prod_{m=1}^{K^{\mathrm{II}}} \frac{\mathscr{X}_0^{0,0}(q,\lambda_m)}{\mathscr{Y}_{0;1}^{0,0;1}(q,\lambda_m)} + \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Z}_{\ell_0;1}^{\ell_0,0;1}(q,p_i) \prod_{m=1}^{K^{\mathrm{II}}} \frac{\mathscr{X}_{\ell_0}^{\ell_0,0}(q,\lambda_m)}{\mathscr{Y}_{\ell_0;1}^{\ell_0,0;1}(q,\lambda_m)} +$$

$$\sum_{k=1}^{\ell_0-1} \prod_{m=1}^{K^{\mathrm{II}}} \frac{\mathscr{X}_k^{k,0}(q,\lambda_m)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_m)} \left\{ \prod_{i=1}^{K^{\mathrm{I}}} \lambda_+(q,p_i) + \prod_{i=1}^{K^{\mathrm{I}}} \lambda_-(q,p_i) \right\} +$$

$$-\sum_{k=0}^{\ell_0-1} \prod_{m=1}^{K^{\mathrm{II}}} \frac{\mathscr{X}_k^{k,0}(q,\lambda_m)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_m)} \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Y}_{k;1}^{k,0;1}(q,p_i) \Lambda^{(6v)} \left(u_0 + \frac{\ell_0 - 1}{2} - k, \vec{u}_\lambda \right), \quad (3.42)$$

where again $\Lambda^{(6v)}$ is the eigenvalue of the auxiliary 6-vertex model, and $\vec{u}_\lambda = (u_{\lambda_1}, \cdots, u_{\lambda_{K^{\mathrm{II}}}})$. The auxiliary roots satisfy the following equations

$$\Lambda^{(6v)}(u_{\lambda_j}, \vec{u}_\lambda) \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Y}_{0;1}^{0,0;1}(\lambda_j, p_i) = 1, \tag{3.43}$$

$$\prod_{i=1}^{K^{\mathrm{II}}} b(w_j, u_{\lambda_i}) \prod_{i=1, i\neq j}^{K^{\mathrm{III}}} \frac{b(w_i, w_j)}{b(w_j, w_i)} = 1. \tag{3.44}$$

In appendix C we give a full derivation of this eigenvalue and auxiliary equations for the case $K^{\mathrm{III}} = 0$. We would also like to mention that the form of the eigenvalues appears in the form of factorized products of single-excitation terms — a somewhat expected feature, which makes us more confident about the generalization procedure.

We would like to point out that the dependence of the auxiliary parameters λ_m only appears in the form $x^+(\lambda_m)$. In order to compare with the known Bethe equations we relabel this to be $x^+(\lambda_m) \equiv y_m$. We also rescale $w_i \longrightarrow \frac{2i}{g} w_i$. In terms of these parameters, the eigenvalues become

$$\Lambda(q|\vec{p}) = \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \tag{3.45}$$

$$+ \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2i\ell_0}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^- x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathscr{X}_{\ell_0}^{\ell_0,0} \right]$$

$$+ \sum_{k=1}^{\ell_0-1} \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \left\{ \prod_{i=1}^{K^{\mathrm{I}}} \lambda_+(q,p_i,k) + \prod_{i=1}^{K^{\mathrm{I}}} \lambda_-(q,p_i,k) \right\}$$

$$- \sum_{k=0}^{\ell_0-1} \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2ik}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \left[1 - \frac{k}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} \right] \times$$

$$\times \mathscr{X}_k^{k,0} \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k-1)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} + \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2ik}{g}}{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2i(k+1)}{g}} \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+3)}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i(2k+1)}{g}} \right\}.$$

and the above auxiliary Bethe equations transform into the well-known ones:

$$\prod_{i=1}^{K^{\text{I}}} \frac{y_k - x_i^-}{y_k - x_i^+} \sqrt{\frac{x_i^+}{x_i^-}} = \prod_{i=1}^{K^{\text{III}}} \frac{w_i - y_k - \frac{1}{y_k} - \frac{i}{g}}{w_i - y_k - \frac{1}{y_k} + \frac{i}{g}}, \tag{3.46}$$

$$\prod_{i=1}^{K^{\text{II}}} \frac{w_k - y_i - \frac{1}{y_i} + \frac{i}{g}}{w_k - y_i - \frac{1}{y_i} - \frac{i}{g}} = \prod_{i=1,i\neq k}^{K^{\text{III}}} \frac{w_k - w_i + \frac{2i}{g}}{w_k - w_i - \frac{2i}{g}}.$$

Once again, we find that for *all* fundamental representations (including the auxiliary space) this agrees with what obtained in [36]. Analogously to formula (41) from the same paper, one can derive the complete set of Bethe equations from the transfer matrix. One finds that the one-particle momenta should satisfy

$$e^{ip_j L} = \Lambda(p_j | \vec{p}). \tag{3.47}$$

The first thing one should notice is that if $q = p_j$ and $\ell_0 = \ell_j$, then $\mathscr{X}_k^{k,0} = 0$ if $k > 0$. This means that the only surviving terms is found to be the first one. This gives the following Bethe equations (after explicitly including the appropriate scalar factor S_0)

$$e^{ip_j L} = \prod_{i=1,i\neq j}^{K^{\text{I}}} S_0(p_j, p_i) \prod_{m=1}^{K^{\text{II}}} \frac{y_m - x_j^-}{y_m - x_j^+} \sqrt{\frac{x_j^+}{x_j^-}}. \tag{3.48}$$

Together with the above set of auxiliary Bethe equations, these indeed agree with the fused Bethe equations.

4 Different vacua and fusion

In the previous sections we deduced the spectrum of the transfer matrix. We found all of its eigenvalues, characterized by the numbers $K^{\text{I,II,III}}$. The eigenvalues were obtained by starting with a vacuum with numbers $K^{\text{II}} = K^{\text{III}} = 0$, which proved to be an eigenstate, and then applying creation operators that generate eigenstates with different quantum numbers. Of course, our choice of vacuum is not unique. We can build up our algebraic Bethe ansatz starting from a different vacuum. One trivial example of this would be to start with w_2 instead of w_1. A more interesting case arises when all physical particles are fermions.

4.1 $\mathfrak{sl}(2)$ vacuum

Consider a fermionic vacuum with all the physical particles in the fundamental representation:

$$|0\rangle_P' = \theta_3 \otimes \ldots \otimes \theta_3. \tag{4.1}$$

This vacuum has quantum numbers $K^{\text{II}} = K^{\text{I}}$ and $K^{\text{III}} = 0$. One can easily check that this vacuum is also an eigenstate. The action of the diagonal elements of fermionic type of

the transfer matrix is given by:

$$\mathcal{T}_{3;k}^{3;k}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{X}_k^{k,0}(q,p_i)|0\rangle'_P,$$

$$\mathcal{T}_{4;k}^{4;k}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{L}_{k;6}^{k,0;6}(q,p_i)|0\rangle'_P.$$

The explicit values for these scattering elements is given in appendix A, and one obtains

$$\mathcal{T}_{3;k}^{3;k}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-}\sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}}|0\rangle'_P, \tag{4.2}$$

$$\mathcal{T}_{4;k}^{4;k}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-}\frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}}\sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}}|0\rangle'_P.$$

Notice that these elements are *independent* of k. This means that, when summing over k, this will only give a factor of ℓ_0.

The next step is to consider the bosonic elements $\mathcal{T}_k^k, \mathcal{T}_{34,k}^{34,k}$. Let us again split off the contributions from $k=0$ and $k=\ell_0$. The corresponding elements $\mathcal{T}_0^0, \mathcal{T}_{\ell_0}^{\ell_0}$ act on this new vacuum as

$$\mathcal{T}_0^0|0\rangle'_P = \mathcal{T}_{\ell_0}^{\ell_0}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Y}_{k;2}^{k,0;2}|0\rangle'_P,$$

$$= \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-}\sqrt{\frac{x_0^+}{x_0^-}}|0\rangle'_P. \tag{4.3}$$

For the remaining elements one finds again, as in the case of the vacuum we have been using before, an additional matrix structure. More precisely, this time one needs to compute the eigenvalues of the matrix

$$\begin{pmatrix} \mathscr{Y}_{k;2}^{k,0;2} & \mathscr{Y}_{k;2}^{k,0;4} \\ \mathscr{Y}_{k;4}^{k,0;2} & \mathscr{Y}_{k;4}^{k,0;4} \end{pmatrix}. \tag{4.4}$$

However, here one encounters the remarkable fact that $\mathscr{Y}_{k;2}^{k,0;4} = \mathscr{Y}_{k;4}^{k,0;2} = 0$, and the matrix is therefore already diagonal. Hence, the eigenvalues are easily read off, and one finds

$$\mathcal{T}_k^k|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Y}_{k;2}^{k,0;2}(q,p_i)|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-}\sqrt{\frac{x_0^+}{x_0^-}}|0\rangle'_P \tag{4.5}$$

and

$$\mathcal{T}_{\alpha4,k}^{\alpha4,k}|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \mathscr{Y}_{k;4}^{k,0;4}(q,p_i)|0\rangle'_P = \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-}\frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}}\sqrt{\frac{x_0^+}{x_0^-}}|0\rangle'_P. \tag{4.6}$$

Similarly to the fermionic contributions (4.2), and once again in contrast to the bosonic vacuum, one finds that these terms are *independent* of k. Summing everything up finally

gives that $|0\rangle'_P$ is an eigenvalue of the transfer matrix with eigenvalue

$$\Lambda(q|\vec{p}) = (\ell_0+1) \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-}} - \ell_0 \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}} - \tag{4.7}$$

$$-\ell_0 \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^-}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} + (\ell_0-1) \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \frac{x_i^- - \frac{1}{x_0^+}}{x_i^+ - \frac{1}{x_0^+}} \sqrt{\frac{x_0^+}{x_0^-}}.$$

This precisely agrees with the result of [38] for antisymmetric representations.

Let us remark that the spectrum is clearly independent of the choice of vacuum. Hence, one should find the same eigenvalues when starting from this or from the bosonic vacuum, that we used in this paper, provided one excites the appropriate set of auxiliary roots. In particular, in the bosonic vacuum one has first to solve the K^{II} auxiliary BAE, and then use these solutions to find the corresponding eigenvalue, which should therefore agree with (4.7). In fact, conversion of one eigenvalue into the other can be obtained by means of duality transformations [40]. We would also like to notice that the result obtained in this section for fundamental representations in the physical space happens to have nice fusion properties, and one can think of combining several of such elementary transfer matrices to obtain more general ones. This approach has been followed for instance in [51].

4.2 $\mathfrak{su}(2)$ vacuum

Let us now come back to the bosonic vacuum we have been using throughout the paper in the derivation of the ABA. In [38], a prescription for computing the transfer matrix eigenvalues on the $\mathfrak{su}(2)$ vacuum (symmetric representation), for all physical legs in the fundamental representation, was also given. The formula was expressed in terms of an expansion of the inverse of a quantum characteristic function. We have found that this prescription indeed produces the same eigenvalues as obtained from our general formula (3.45), when restricting the latter to fundamental particles in the physical space. To this purpose, we explicitly work out here below the above mentioned expansion following [38], adapting the calculation to the notations we use in this paper. We will then compare the final formula with the suitable restriction of our result (3.45), finding perfect agreement. Indeed, we will be able to relax the condition of physical legs in the fundamental representation, by making the conjectured expression for the quantum characteristic function slightly more general. We will then find agreement with such a formula in the general case $\ell_i \neq 1$ as well. All these agreements will however be reached in a quite non-trivial and interesting fashion. We therefore think that the explicit calculation we reproduce in what follows will precisely help clarifying this fact.

Following [38], we define the shift operator U by

$$U f(u) U^{-1} = f\left(u + \frac{1}{2}\right), \tag{4.8}$$

and introduce the notation

$$f^{[\ell]}(u) \equiv U^\ell f(u) U^{-\ell} = f\left(u + \frac{\ell}{2}\right). \tag{4.9}$$

The spectral parameters of an elementary particle, defined in (3.6), satisfy the relation

$$x^{[1]} + \frac{1}{x^{[1]}} - x^{[-1]} - \frac{1}{x^{[-1]}} = \frac{2i}{g} \, . \tag{4.10}$$

By successive applications of the shift operator to (4.10), one finds that the pair of variables $\{x^{[\ell]}, x^{[\ell-2k]}\}$ defines another rapidity torus

$$x^{[\ell]} + \frac{1}{x^{[\ell]}} - x^{[\ell-2k]} - \frac{1}{x^{[\ell-2k]}} = \frac{2ik}{g} \, . \tag{4.11}$$

There are two choices of branch for $x_a^{[\ell-2k]}$ for a given $x^{[\ell]}$, as can be seen by

$$x^{[\ell-2k]} = \frac{1}{2} \left(x^{[\ell]} + \frac{1}{x^{[\ell]}} - \frac{2ik}{g} + \sqrt{\left(x^{[\ell]} + \frac{1}{x^{[\ell]}} - \frac{2ik}{g} \right)^2 - 4} \right) . \tag{4.12}$$

We also use $y_i + 1/y_i = iv_i$ in what follows.[10]

Let $\langle \ell_0 - 1, 0 \rangle$ be the ℓ_0-th symmetric representation of $\mathfrak{su}(2|2)$. The conjecture states that the transfer matrix for such a representation $T_{\langle \ell_0-1,0 \rangle}(u_0|\{\vec{u}, \vec{v}, \vec{w}\})$ is generated by $T_{\langle 0,0 \rangle}(u_0|\{\vec{u}, \vec{v}, \vec{w}\})$, where the generating function is equal to the inverse of the quantum characteristic function:

$$\begin{aligned} D_0^{-1} &:= \left(1 - U_0 T_4 U_0 \right)^{-1} \left(1 - U_0 T_3 U_0 \right) \left(1 - U_0 T_2 U_0 \right) \left(1 - U_0 T_1 U_0 \right)^{-1}, \tag{4.13} \\ &= \left(1 + \sum_{h=1}^{\infty} (U_0 T_4 U_0)^h \right) \left(1 - U_0 T_3 U_0 \right) \left(1 - U_0 T_2 U_0 \right) \left(1 + \sum_{k=1}^{\infty} (U_0 T_1 U_0)^k \right), \\ &\equiv \sum_{\ell_0=0}^{\infty} U_0^{\ell_0} \, T_{\langle \ell_0-1,0 \rangle}(u_0|\{\vec{u}, \vec{v}, \vec{w}\}) \, U_0^{\ell_0} \, . \tag{4.14} \end{aligned}$$

Here U_0 is the shift operator for u_0. The first few terms can be found as follows:

$$\begin{aligned} D_0^{-1} = \ &1 + U_0 \left(T_4 - T_3 - T_2 + T_1 \right) U_0 \tag{4.15} \\ &+ U_0^2 \Big\{ T_4^{[-1]} T_4^{[1]} + T_4^{[-1]} T_1^{[1]} + T_1^{[-1]} T_1^{[1]} + T_3^{[-1]} T_2^{[1]} \\ &\quad - T_4^{[-1]}(T_3^{[1]} + T_2^{[1]}) - (T_3^{[-1]} + T_2^{[-1]}) T_1^{[1]} \Big\} U_0^2 \\ &+ U_0^3 \Big\{ T_4^{[-2]} T_4 T_4^{[+2]} + T_4^{[-2]} T_4 T_1^{[+2]} + T_4^{[-2]} T_1 T_1^{[+2]} + T_1^{[-2]} T_1 T_1^{[+2]} + T_4^{[-2]} T_3 T_2^{[+2]} \\ &\quad + T_3^{[-2]} T_2 T_1^{[+2]} - T_4^{[-2]} T_4 \left(T_3^{[+2]} + T_2^{[+2]} \right) - T_4^{[-2]} \left(T_3 + T_2 \right) T_1^{[+2]} \\ &\quad - \left(T_3^{[-2]} + T_2^{[-2]} \right) T_1 T_1^{[+2]} \Big\} U_0^3 \ + \ \cdots, \tag{4.16} \end{aligned}$$

and, in general,

$$T_{\langle \ell_0-1,0 \rangle}(u_0|\{\vec{u}, \vec{v}, \vec{w}\}) = \tau_{\ell_0,0} - \tau_{\ell_0,1} \left[T_3 \right] - \tau_{\ell_0,1} \left[T_2 \right] + \tau_{\ell_0,2} \left[T_3, T_2 \right], \tag{4.17}$$

---

[10]Interestingly, the final result (4.45) is almost invariant under the map $y_i \mapsto 1/y_i$, except for an overall factor.

where

$$\tau_{\ell_0,0} = \sum_{k=0}^{\ell_0} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \cdots T_4^{[\ell_0-2k-3]} T_1^{[\ell_0-2k-1]} \cdots T_1^{[\ell_0-1]}, \qquad (4.18)$$

$$\tau_{\ell_0,1}[X] = \sum_{k=0}^{\ell_0-1} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \cdots T_4^{[-\ell_0+2k-1]} \times$$
$$X^{[\ell_0-2k-1]} T_1^{[\ell_0-2k+1]} \cdots T_1^{[\ell_0-1]}, \qquad (4.19)$$

$$\tau_{\ell_0,2}[X,Y] = \sum_{k=0}^{\ell_0-2} T_4^{[-\ell_0+1]} T_4^{[-\ell_0+3]} \cdots T_4^{[-\ell_0+2k-1]} \times$$
$$X^{[\ell_0-2k-3]} Y^{[\ell_0-2k-1]} T_1^{[\ell_0-2k+1]} \cdots T_1^{[\ell_0-1]}. \qquad (4.20)$$

The first line of (4.15) gives the transfer matrix for the fundamental representation as

$$T_{\langle 0,0 \rangle}(u_0|\{\vec{u},\vec{v},\vec{w}\}) = T_1 - T_2 - T_3 + T_4. \qquad (4.21)$$

We recall that the left hand side of this equation is given explicitly by (3.45) at $\ell_0 = 1$, which reads

$$\Lambda(q|\vec{p}) = \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} + \qquad (4.22)$$
$$+ \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i}}{x_0^+ + \frac{1}{x_0^+} - y_i - \frac{1}{y_i} - \frac{2i}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{(x_0^- - x_i^-)(1 - x_0^- x_i^+)}{(x_0^- - x_i^+)(1 - x_0^+ x_i^+)} \sqrt{\frac{x_0^+ x_i^+}{x_0^- x_i^-}} \mathscr{X}_1^{1,0} \right]$$
$$- \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^- - x_i^+} \sqrt{\frac{x_0^-}{x_0^+}} \times$$
$$\times \mathscr{X}_0^{0,0} \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} - \frac{i}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i}{g}} + \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+}}{y_i + \frac{1}{y_i} - x_0^+ - \frac{1}{x_0^+} + \frac{2i}{g}} \prod_{i=1}^{K^{\mathrm{III}}} \frac{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{3i}{g}}{w_i - x_0^+ - \frac{1}{x_0^+} + \frac{i}{g}} \right\}.$$

Therefore, $\Lambda(q|\vec{p})$ may be equated with the right hand side of (4.21) term by term. We simplify the above expression of $\Lambda(q|\vec{p})$ by introducing variables w_i and v_i as follows:[11]

$$w_i - x_0^\pm - \frac{1}{x_0^\pm} + \frac{in}{g} \equiv \left(\mathsf{w}_i - u_0 + \frac{\mp \ell_0 + n}{2} \right) \frac{2i}{g}, \qquad (4.23)$$

$$y_i + \frac{1}{y_i} - x_0^\pm - \frac{1}{x_0^\pm} + \frac{in}{g} \equiv \left(\mathsf{v}_i - u_0 + \frac{\mp \ell_0 + n}{2} \right) \frac{2i}{g}. \qquad (4.24)$$

With the help of (3.5) and

$$\mathscr{X}_1^{1,0} = \frac{u_0 - u_i + \frac{1-\ell_i}{2}}{u_0 - u_i + \frac{\ell_i-1}{2}} \mathcal{D} = \frac{\left(x_0^+ - x_i^+\right) \left(1 - \frac{1}{x_0^+ x_i^+}\right) \left(x_0^- - x_i^+\right)}{\left(x_0^- - x_i^-\right) \left(1 - \frac{1}{x_0^- x_i^-}\right) \left(x_0^+ - x_i^-\right)} \sqrt{\frac{x_0^+ x_i^-}{x_0^- x_i^+}}, \qquad (4.25)$$

---

[11]Our notation is $x_0^\pm = x^{[\pm \ell_0]}$, and $\ell_0 = 1$ is used when discussing the fundamental transfer matrix. Note that the shift operator does not act on $x_i^\pm$.

it produces

$$\Lambda(q|\vec{p}) = \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \tag{4.26}$$

$$+ \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \left[\frac{\mathsf{v}_i - u_0 - \frac{1}{2}}{\mathsf{v}_i - u_0 + \frac{1}{2}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \frac{(x_0^+ - x_i^+)\left(1 - \frac{1}{x_0^- x_i^+}\right)}{(x_0^+ - x_i^-)\left(1 - \frac{1}{x_0^- x_i^-}\right)}$$

$$- \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}} \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \times$$

$$\times \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 - 1}{\mathsf{w}_i - u_0} + \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{1}{2}}{\mathsf{v}_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 + 1}{\mathsf{w}_i - u_0} \right\}.$$

It is useful to separate a common factor in the following fashion:

$$T_j = S_{\langle 0,0 \rangle} \tilde{T}_j, \qquad S_{\langle 0,0 \rangle} \equiv \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^-}{y_i - x_0^+} \sqrt{\frac{x_0^+}{x_0^-}}, \qquad (j = 1, \ldots, 4). \tag{4.27}$$

Then, the tilded functions can be written as

$$\tilde{T}_1 = \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{1}{2}}{\mathsf{v}_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \frac{\left(1 - \frac{1}{x_0^- x_i^+}\right)(x_0^+ - x_i^+)}{\left(1 - \frac{1}{x_0^- x_i^-}\right)(x_0^+ - x_i^-)}, \tag{4.28}$$

$$\tilde{T}_2 = \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 + 1}{\mathsf{w}_i - u_0} \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{1}{2}}{\mathsf{v}_i - u_0 + \frac{1}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}}, \tag{4.29}$$

$$\tilde{T}_3 = \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 - 1}{\mathsf{w}_i - u_0} \prod_{i=1}^{K^{\mathrm{I}}} \frac{x_0^+ - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}}, \tag{4.30}$$

$$\tilde{T}_4 = 1. \tag{4.31}$$

Note that different identification of $\tilde{T}_i$'s would produce the transfer matrix for different representations [40].

Let us evaluate the function τ's appearing in the conjectured transfer matrix for the ℓ_0-th symmetric representation (4.17). They can be simplified by using

$$\tilde{T}_1^{[\ell_0-1]} = \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{1 - \frac{1}{x_0^{[\ell_0-2]} x_i^+}}{1 - \frac{1}{x_0^{[\ell_0-2]} x_i^-}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \right],$$

$$= \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[\ell_0-2]} - x_i^-}{x_0^{[\ell_0-2]} - x_i^+} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2}{2}}{u_0 - u_i + \frac{\ell_0+\ell_i-2}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \right],$$

$$= \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2}{2}} \times$$

$$\prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[\ell_0-2]} - x_i^-}{x_0^{[\ell_0-2]} - x_i^+} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2}{2}}{u_0 - u_i + \frac{\ell_0+\ell_i-2}{2}} \frac{u_0 - u_i + \frac{\ell_0-\ell_i}{2}}{u_0 - u_i + \frac{\ell_0+\ell_i}{2}} \frac{1 - \frac{1}{x_0^{[\ell_0]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \right] \tag{4.32}$$

and, therefore,

$$\tilde{T}_1^{[\ell_0-3]}\,\tilde{T}_1^{[\ell_0-1]} = \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-4}{2}} \;\times$$

$$\prod_{i=1}^{K^{\mathrm{I}}}\left[\frac{x_0^{[\ell_0-4]}-x_i^-}{x_0^{[\ell_0-4]}-x_i^+}\,\frac{u_0-u_i+\frac{\ell_0-\ell_i-4}{2}}{u_0-u_i+\frac{\ell_0+\ell_i-4}{2}}\,\frac{u_0-u_i+\frac{\ell_0-\ell_i-2}{2}}{u_0-u_i+\frac{\ell_0+\ell_i-2}{2}}\,\frac{x_0^{[\ell_0]}-x_i^+}{x_0^{[\ell_0]}-x_i^-}\right]. \quad (4.33)$$

Thus, $\tau_{\ell_0,0}$ becomes

$$\tau_{\ell_0,0} = S_{\langle\ell_0-1,0\rangle}\;\times \hspace{7cm} (4.34)$$
$$\left(1+\sum_{k=1}^{\ell_0}\left\{\prod_{i=1}^{K^{\mathrm{II}}}\frac{\mathsf{v}_i-u_0-\frac{\ell_0}{2}}{\mathsf{v}_i-u_0-\frac{\ell_0-2k}{2}}\prod_{i=1}^{K^{\mathrm{I}}}\left[\frac{x_0^{[\ell_0-2k]}-x_i^-}{x_0^{[\ell_0-2k]}-x_i^+}X_1^k\left(u_0-u_i\,,\ell_0\,,\ell_i\right)\frac{x_0^{[\ell_0]}-x_i^+}{x_0^{[\ell_0]}-x_i^-}\right]\right\}\right),$$

where we have introduced

$$S_{\langle\ell_0-1,0\rangle} = \prod_{k=1}^{\ell_0}S_{\langle 0,0\rangle}^{[\ell_0+1-2k]} = \prod_{i=1}^{K^{\mathrm{II}}}\frac{y_i-x_0^{[-\ell_0]}}{y_i-x_0^{[\ell_0]}}\sqrt{\frac{x_0^{[\ell_0]}}{x_0^{[-\ell_0]}}}\,, \quad (4.35)$$

$$X_l^k\left(u,\ell_0\,,\ell_i\right) = \prod_{j=l}^{k}\frac{u+\frac{\ell_0-\ell_i-2j}{2}}{u+\frac{\ell_0+\ell_i-2j}{2}}\,. \quad (4.36)$$

Similarly, $\tau_{\ell_0,1}\,[T_3]\,,\tau_{\ell_0,1}\,[T_2]$ and $\tau_{\ell_0,2}\,[T_3\,,T_2]$ are given by

$$\tau_{\ell_0,1}\,[T_3] = S_{\langle\ell_0-1,0\rangle}\sum_{k=0}^{\ell_0-1}\left\{\prod_{i=1}^{K^{\mathrm{III}}}\frac{\mathsf{w}_i-u_0-\frac{\ell_0-2k+1}{2}}{\mathsf{w}_i-u_0-\frac{\ell_0-2k-1}{2}}\prod_{i=1}^{K^{\mathrm{II}}}\frac{\mathsf{v}_i-u_0-\frac{\ell_0}{2}}{\mathsf{v}_i-u_0-\frac{\ell_0-2k}{2}}\;\times \quad (4.37)\right.$$
$$\left.\prod_{i=1}^{K^{\mathrm{I}}}\left[\sqrt{\frac{x_i^-}{x_i^+}}X_1^k\left(u_0-u_i,\ell_0\,,\ell_i\right)\frac{x_0^{[\ell_0]}-x_i^+}{x_0^{[\ell_0]}-x_i^-}\right]\right\},$$

$$\tau_{\ell_0,1}\,[T_2] = S_{\langle\ell_0-1,0\rangle}\sum_{k=0}^{\ell_0-1}\left\{\prod_{i=1}^{K^{\mathrm{III}}}\frac{\mathsf{w}_i-u_0-\frac{\ell_0-2k-3}{2}}{\mathsf{w}_i-u_0-\frac{\ell_0-2k-1}{2}}\prod_{i=1}^{K^{\mathrm{II}}}\frac{\mathsf{v}_i-u_0-\frac{\ell_0}{2}}{\mathsf{v}_i-u_0-\frac{\ell_0-2k-2}{2}}\;\times \quad (4.38)\right.$$
$$\left.\prod_{i=1}^{K^{\mathrm{I}}}\left[\sqrt{\frac{x_i^-}{x_i^+}}X_1^k\left(u_0-u_i,\ell_0\,,\ell_i\right)\frac{x_0^{[\ell_0]}-x_i^+}{x_0^{[\ell_0]}-x_i^-}\right]\right\},$$

$$\tau_{\ell_0,2}\,[T_3\,,T_2] = S_{\langle\ell_0-1,0\rangle}\sum_{k=0}^{\ell_0-2}\left\{\prod_{i=1}^{K^{\mathrm{II}}}\frac{\mathsf{v}_i-u_0-\frac{\ell_0}{2}}{\mathsf{v}_i-u_0-\frac{\ell_0-2k-2}{2}}\;\times \quad (4.39)\right.$$
$$\left.\prod_{i=1}^{K^{\mathrm{I}}}\left[\frac{x_i^-}{x_i^+}\,\frac{1-\frac{1}{x_0^{[\ell_0-2k-2]}x_i^-}}{1-\frac{1}{x_0^{[\ell_0-2k-2]}x_i^+}}X_1^{k+1}\left(u_0-u_i,\ell_0\,,\ell_i\right)\frac{x_0^{[\ell_0]}-x_i^+}{x_0^{[\ell_0]}-x_i^-}\right]\right\}.$$

The functions $\tau_{\ell_0,0}$ and $\tau_{\ell_0,2}[T_3, T_2]$ can be combined together as

$$\tau_{\ell_0,0} + \tau_{\ell_0,2}[T_3, T_2] = S_{\langle \ell_0-1,0 \rangle} \; \times \qquad\qquad (4.40)$$

$$\left(1 + \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 + \frac{\ell_0}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[-\ell_0]} - x_i^+} X_0^{\ell_0}(u_0 - u_i, \ell_0, \ell_i) \frac{1 - \frac{1}{x_0^{[\ell_0]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \right] \right.$$

$$+ \sum_{k=1}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[\ell_0-2k]} - x_i^-}{x_0^{[\ell_0-2k]} - x_i^+} X_1^k(u_0 - u_i, \ell_0, \ell_i) \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \right] \right\}$$

$$\left. + \sum_{k=1}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_i^-}{x_i^+} \frac{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^+}} X_1^k(u_0 - u_i, \ell_0, \ell_i) \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \right] \right\} \right).$$

Let us compare the transfer matrix for the ℓ_0-th symmetric representation (3.45) with the conjectured one (4.17). Consider the fermionic terms first. The fourth and fifth lines of (3.45) should be compared with $\tau_{\ell_0,1}[T_3] + \tau_{\ell_0,1}[T_2]$ in (4.37) and (4.38). From (3.5) one can deduce the identity[12]

$$\frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2k}{2}}{u_0 - u_i + \frac{\ell_0 - \ell_i}{2}} = \prod_{j=1}^{k} \frac{u_0 - u_i + \frac{\ell_0 - \ell_i - 2j}{2}}{u_0 - u_i + \frac{\ell_0 + \ell_i - 2j}{2}} = X_1^k(u_0 - u_i, \ell_0, \ell_i). \qquad (4.41)$$

By using the relation

$$x_0^{[\pm \ell_0]} = x_0^{\pm}, \qquad \mathcal{D} = \frac{x_0^- - x_i^+}{x_0^+ - x_i^-} \sqrt{\frac{x_0^+}{x_0^-} \frac{x_i^-}{x_i^+}}, \qquad (4.42)$$

one finds nice agreement for the fermionic terms.

Next, let us look at the bosonic terms. The first, second, and third lines of (3.45) should be compared with $\tau_{\ell_0} + \tau_{\ell_0,2}[T_3, T_2]$ in (4.40). The first term is $S_{\langle \ell_0-1,0 \rangle}$ for both. The second term also agrees because of the relation

$$\frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[-\ell_0]} - x_i^+} X_0^{\ell_0}(u_0 - u_i, \ell_0, \ell_i) \frac{1 - \frac{1}{x_0^{[\ell_0]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}}$$

$$= \frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[\ell_0]} - x_i^-} \frac{1 - \frac{1}{x_0^{[-\ell_0]} x_i^+}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \frac{u_0 - u_i + \frac{\ell_0 + \ell_i}{2}}{u_0 - u_i - \frac{\ell_0 + \ell_i}{2}} X_0^{\ell_0}(u_0 - u_i, \ell_0, \ell_i),$$

$$= \frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[\ell_0]} - x_i^-} \frac{1 - \frac{1}{x_0^{[-\ell_0]} x_i^+}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \frac{\mathscr{X}_{\ell_0}^{\ell_0,0}}{\mathcal{D}}. \qquad (4.43)$$

Thus, we can proceed to identify the rest of the bosonic terms, namely the last two lines of the equation (4.40) and the third line of (3.45). By substituting $x^{[\ell_0-2k]}$ of (4.12) into

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[12]It should be noted that $\mathcal{D}$ and $\mathscr{X}_k^{k,0}$ actually depend on i, and that $\mathcal{D} = \mathscr{X}_k^{k,0}$ when $\ell_i = 1$ for each i.

the definition of $\lambda_\pm(q, p_i, k)$ in (3.20), we find

$$\lambda_\pm(q, p_i, k) = \begin{cases} \dfrac{x_0^{[\ell_0-2k]} - x_i^-}{x_0^{[\ell_0-2k]} - x_i^+} \dfrac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \dfrac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \dfrac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \,, \\[3ex] \dfrac{x_i^-}{x_i^+} \dfrac{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^+}} \dfrac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \dfrac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \dfrac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \,. \end{cases} \tag{4.44}$$

Thanks to the previous identity (4.41), the rest of the bosonic terms of both turn out to be identical. This completes the proof.

For the reader's convenience, we rewrite here the transfer matrix for the ℓ_0-th symmetric representation, obtained from the conjecture on the quantum characteristic function:

$$T_{\langle \ell_0-1, 0 \rangle}(u_0 | \{\vec{u}, \vec{v}, \vec{w}\}) = \prod_{i=1}^{K^{\mathrm{II}}} \frac{y_i - x_0^{[-\ell_0]}}{y_i - x_0^{[\ell_0]}} \sqrt{\frac{x_0^{[\ell_0]}}{x_0^{[-\ell_0]}}} \times \tag{4.45}$$

$$\left(1 + \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 + \frac{\ell_0}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[-\ell_0]} - x_i^-}{x_0^{[\ell_0]} - x_i^-} \frac{1 - \frac{1}{x_0^{[-\ell_0]} x_i^+}}{1 - \frac{1}{x_0^{[\ell_0]} x_i^+}} \frac{\mathscr{X}_{\ell_0}^{\ell_0,0}}{\mathcal{D}} \right] \right.$$

$$+ \sum_{k=1}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k}{2}} \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_0^{[\ell_0-2k]} - x_i^-}{x_0^{[\ell_0-2k]} - x_i^+} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \right] \right\}$$

$$+ \sum_{k=1}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k}{2}} \right.$$

$$\left. \times \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{x_i^-}{x_i^+} \frac{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^-}}{1 - \frac{1}{x_0^{[\ell_0-2k]} x_i^+}} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \right] \right\}$$

$$- \sum_{k=0}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 - \frac{\ell_0-2k+1}{2}}{\mathsf{w}_i - u_0 - \frac{\ell_0-2k-1}{2}} \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k}{2}} \right.$$

$$\left. \times \prod_{i=1}^{K^{\mathrm{I}}} \left[\sqrt{\frac{x_i^-}{x_i^+}} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \right] \right\}$$

$$- \sum_{k=0}^{\ell_0-1} \left\{ \prod_{i=1}^{K^{\mathrm{III}}} \frac{\mathsf{w}_i - u_0 - \frac{\ell_0-2k-3}{2}}{\mathsf{w}_i - u_0 - \frac{\ell_0-2k-1}{2}} \prod_{i=1}^{K^{\mathrm{II}}} \frac{\mathsf{v}_i - u_0 - \frac{\ell_0}{2}}{\mathsf{v}_i - u_0 - \frac{\ell_0-2k-2}{2}} \right.$$

$$\left. \left. \times \prod_{i=1}^{K^{\mathrm{I}}} \left[\sqrt{\frac{x_i^-}{x_i^+}} \frac{u_0 - u_i + \frac{\ell_0-\ell_i-2k}{2}}{u_0 - u_i + \frac{\ell_0-\ell_i}{2}} \frac{x_0^{[\ell_0]} - x_i^+}{x_0^{[\ell_0]} - x_i^-} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \right] \right\} \right) \,.$$

How the agreement works can be understood in the following way. From the expression (4.45) we see that, apparently, a spurious dependence on the parameters $x_0^{[\ell_0-2k]}$ is left as a remnant of the fusion among the different blocks of the quantum characteristic function. However, one can make use of (4.12) to re-express each of these variables only in terms of the bound state variable $x_0^{[\ell_0]}$, provided one chooses a branch of the quadratic map. The remarkable observation is that, after this replacement, one can recast the above expression in a form that precisely agrees with our result (3.45). This happens for both

choices of branch, consistent with the fact that the formula we have obtained *via* the alternative route of the ABA does not bear any dependence on such a choice.

The transfer matrix for the symmetric representations (4.45) without auxiliary roots can be used to compute the wrapping correction in the $\mathfrak{sl}(2)$ sector after the simultaneous flip $(x_0^+, x_i^+) \leftrightarrow (x_0^-, x_i^-)$. By plugging the appropriate solution of the asymptotic Bethe Ansatz equation into (4.45), one can reproduce the results obtained by [19, 25].

Acknowledgments

A Elements of the bound state scattering matrix

In this appendix we list the elements of the bound state S-matrix from [35] that are used in this paper. We start with the Case I S-matrix coefficients[13]

$$
\begin{aligned}
\mathscr{X}_n^{k,l} = (-1)^{k+n}\, \pi \mathcal{D} & \frac{\sin[(k-\ell_1)\pi]\,\Gamma(l+1)}{\sin[\ell_1\pi]\sin[(k+l-\ell_2-n)\pi]\,\Gamma(l-\ell_2+1)\Gamma(n+1)} \times \\
& \frac{\Gamma(n+1-\ell_1)\Gamma\left(l+\frac{\ell_1-\ell_2}{2}-n-\delta u\right)\Gamma\left(1-\frac{\ell_1+\ell_2}{2}-\delta u\right)}{\Gamma\left(k+l-\frac{\ell_1+\ell_2}{2}-\delta u+1\right)\Gamma\left(\frac{\ell_1-\ell_2}{2}-\delta u\right)} \times \\
& {}_4\tilde{F}_3\left(-k,-n,\delta u+1-\frac{\ell_1-\ell_2}{2},\frac{\ell_2-\ell_1}{2}-\delta u;1-\ell_1,\ell_2-k-l,l-n+1;1\right).
\end{aligned}
\tag{A.1}
$$

One has defined ${}_4\tilde{F}_3(x,y,z,t;r,v,w;\tau) = {}_4F_3(x,y,z,t;r,v,w;\tau)/[\Gamma(r)\Gamma(v)\Gamma(w)]$,

$$
\mathcal{D} = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}}
\tag{A.2}
$$

and

$$
\delta u = u_1 - u_2.
$$

The relevant entries of the Case II S-matrix are given by

$$
\mathscr{Y}_{k;1}^{k,0;1} = \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\sqrt{\frac{x_1^-}{x_1^+}}\left[1-\frac{k}{\delta u + \frac{\ell_1-\ell_2}{2}}\right]\mathscr{X}^{k,0},
\tag{A.3}
$$

$$
\mathscr{Y}_{k;2}^{k,0;2} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\sqrt{\frac{x_2^+}{x_2^-}}\,\mathscr{X}^{k,0},
\tag{A.4}
$$

---

[13]We suppress the dependence on momenta in order to have a lighter notation. All functions appearing in this section have to be understood as $\mathscr{X} \equiv \mathscr{X}(p_1,p_2)$, $\mathscr{Y} \equiv \mathscr{Y}(p_1,p_2)$, $\mathscr{Z} \equiv \mathscr{Z}(p_1,p_2)$ (indices are omitted here for simplicity) and $\mathcal{D} = \mathcal{D}(p_1,p_2)$.

$$\mathscr{Y}_{k;2}^{k,0;1} = \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \sqrt{\frac{x_1^- x_2^+}{x_1^+ x_2^-}} \frac{\sqrt{\ell_1}\eta(p_1)}{\sqrt{\ell_2}\eta(p_2)} \frac{k-\ell_1}{\ell_1} \mathscr{X}_k^{k,0}, \tag{A.5}$$

$$\mathscr{Y}_{k;1}^{k,0;2} = \frac{x_1^+ - x_1^-}{x_1^- - x_2^+} \frac{\sqrt{\ell_2}\eta(p_2)}{\sqrt{\ell_1}\eta(p_1)} \mathscr{X}_k^{k,0}, \tag{A.6}$$

$$\mathscr{Y}_{k;1}^{k,0;4} = \frac{\sqrt{\ell_1 \ell_2}\eta(p_1)\eta(p_2)}{x_1^+ x_2^+ - 1} \frac{k}{i\ell_1} \mathscr{X}_k^{k,0}, \tag{A.7}$$

$$\mathscr{Y}_{k;4}^{k,0;4} = \sqrt{\frac{x_2^+}{x_2^-}} \frac{x_1^+ x_2^- - 1}{x_1^+ x_2^+ - 1} \mathscr{X}_k^{k,0}, \tag{A.8}$$

$$\mathscr{Y}_{k;4}^{k,0;1} = \frac{i}{\sqrt{\ell_1 \ell_2}\eta(p_1)\eta(p_2)} \sqrt{\frac{x_1^+ x_2^+}{x_1^- x_2^-}} \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)}{x_1^+ x_2^+ - 1} \mathscr{X}_k^{k,0}, \tag{A.9}$$

Finally, from the Case III S-matrix we used

$$\mathscr{Z}_{k;1}^{k,0;1} = \left[1 - \frac{2ik}{g} \frac{x_1^+(x_2^- - x_1^- x_1^+ x_2^+)}{(x_2^- - x_1^+)(1 - x_1^- x_1^+)(1 - x_1^+ x_2^+)} \right] \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}}, \tag{A.10}$$

$$\mathscr{Z}_{k;3}^{k,0;1} = \frac{2(x_1^- - x_1^+)(x_1^+)^2(x_2^- - x_2^+)}{g(x_1^+ - x_2^-)(1 - x_1^+ x_1^-)(1 - x_1^+ x_2^+)\eta(p_1)^2} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}}, \tag{A.11}$$

$$\mathscr{Z}_{k;1}^{k,0;3} = \frac{ik(\ell_1 - k)}{\ell_1} \frac{(x_2^- - x_2^+)\eta(p_1)^2}{(x_1^+ - x_2^-)(1 - x_1^+ x_2^+)} \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}}, \tag{A.12}$$

$$\mathscr{Z}_{k;3}^{k,0;3} = \left[\frac{(x_1^+ - x_2^+)(1 - x_2^- x_1^+)}{(x_1^+ - x_2^-)(1 - x_1^+ x_2^+)} + \frac{2ik}{g} \frac{x_1^+(x_2^+ - x_1^- x_1^+ x_2^-)}{(x_1^+ - x_2^-)(1 - x_1^- x_1^+)(1 - x_1^+ x_2^+)} \right] \frac{\mathscr{X}_k^{k,0}}{\mathcal{D}} \tag{A.13}$$

and

$$\mathscr{Z}_{k;1}^{k,0;6} = \frac{ik\sqrt{\ell_2}\eta(p_1)\eta(p_2)}{\sqrt{\ell_1}} \frac{x_1^- - x_2^-}{(x_1^- - x_2^+)(1 - x_1^+ x_2^+)} \sqrt{\frac{x_2^+}{x_2^-}} \mathscr{X}_k^{k,0}, \tag{A.14}$$

$$\mathscr{Z}_{k;3}^{k,0;6} = \sqrt{\frac{\ell_2}{\ell_1}} \frac{\eta(p_2)}{\eta(p_1)} \frac{(x_1^- - x_1^+)(x_2^- x_1^+ - 1)}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)} \sqrt{\frac{x_2^+}{x_2^-}} \mathscr{X}_k^{k,0}, \tag{A.15}$$

$$\mathscr{Z}_{k;6}^{k,0;6} = \frac{(x_1^- - x_2^-)(x_2^- x_1^+ - 1)x_2^+}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)x_2^-} \mathscr{X}_k^{k,0}, \tag{A.16}$$

$$\mathscr{Z}_{k;6}^{k,0;3} = \frac{(\ell_1 - k)\eta(p_1)}{\sqrt{\ell_1 \ell_2}\eta(p_2)} \frac{(x_2^- x_1^+ - 1)(x_2^- - x_2^+)x_2^+}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)x_2^-} \sqrt{\frac{x_1^-}{x_1^+}} \mathscr{X}_k^{k,0}, \tag{A.17}$$

$$\mathscr{Z}_{k;6}^{k,0;1} = \frac{i}{\sqrt{\ell_1 \ell_2}\eta(p_1)\eta(p_2)} \frac{(x_1^- - x_2^-)(x_1^- - x_1^+)(x_2^- - x_2^+)x_2^+}{(x_1^- - x_2^+)(x_1^+ x_2^+ - 1)x_2^-} \sqrt{\frac{x_1^+}{x_1^-}} \mathscr{X}_k^{k,0}. \tag{A.18}$$

B   Algebraic Bethe ansatz for the 6-vertex model

In the this paper we use the algebraic Bethe ansatz approach to diagonalize the $AdS_5 \times S^5$ superstring transfer matrix for bound states. We will closely follow the discussion for the Hubbard model [49]. In this model, just as in our case, the 6-vertex model plays an

important role. In this section we will discuss the algebraic Bethe ansatz for this model for completeness and to fix notations.

The algebraic Bethe ansatz for the 6-vertex model is a standard chapter of the theory of integrable systems, and it is treated for example in [48, 52]. The scattering matrix of the model is given by

$$r_{12}(u_1, u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u_1, u_2) & a(u_1, u_2) & 0 \\ 0 & a(u_1, u_2) & b(u_1, u_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.1}$$

where

$$a(u_1, u_2) = \frac{U}{u_1 - u_2 + U}, \qquad b(u_1, u_2) = \frac{u_1 - u_2}{u_1 - u_2 + U} \tag{B.2}$$

It is convenient to write it as

$$\begin{aligned} r_{12}(u_1, u_2) &= r_{\alpha\beta}^{\gamma\delta}(u_1, u_2) E_\gamma^\alpha \otimes E_\delta^\beta \\ &= \frac{u_1 - u_2}{u_1 - u_2 + U} \left[E_\alpha^\alpha \otimes E_\beta^\beta + \frac{U}{u_1 - u_2} E_\beta^\alpha \otimes E_\alpha^\beta \right], \end{aligned} \tag{B.3}$$

with E_β^α the standard matrix unities. Let us consider K particles, with rapidities u_i. Now, one can construct the monodromy matrix

$$\mathcal{T}(u_0|\vec{u}) = \prod_{i=1}^K r_{0i}(u_0|u_i). \tag{B.4}$$

Let us write it as a matrix in the auxiliary space

$$\mathcal{T}^{(1)}(u_0|\vec{u}) = \begin{pmatrix} A(u_0|\vec{u}) & B(u_0|\vec{u}) \\ C(u_0|\vec{u}) & D(u_0|\vec{u}) \end{pmatrix}. \tag{B.5}$$

In the algebraic Bethe Ansatz, one constructs the eigenvalues of the transfer matrix by first specifying a ground state $|0\rangle$. The ground state, in this case, is defined as

$$|0\rangle = \bigotimes_{i=1}^K \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{B.6}$$

It is easily checked that it is an eigenstate of the transfer matrix. To be a bit more precise, the action of the different elements of the monodromy matrix on $|0\rangle$ is given by

$$\begin{aligned} A(u_0|\vec{u})|0\rangle &= |0\rangle, \\ C(u_0|\vec{u})|0\rangle &= 0, \\ D(u_0|\vec{u})|0\rangle &= \prod_{i=1}^K b(u_0, u_i)|0\rangle. \end{aligned} \tag{B.7}$$

Thus, $|0\rangle$ is an eigenstate of the transfer matrix with the following eigenvalue

$$1 + \prod_{i=1}^K b(u_0, u_i). \tag{B.8}$$

The field B will be considered as a creation operator. It will create all the other eigenstates out of the vacuum. We introduce additional parameters w_i and consider the state

$$|M\rangle := \phi_M(w_1, \ldots, w_M)|0\rangle, \qquad \phi_M(w_1, \ldots, w_M) := \prod_{i=1}^{M} B(w_i|\vec{u}). \tag{B.9}$$

In the context of the Heisenberg spin chain the vacuum corresponds to all spins down and the state $|M\rangle$ corresponds to the eigenstate of the transfer matrix that has M spins turned up.

In order to evaluate the action of the transfer matrix $\mathscr{T}(u_0|\vec{u}) = A(u_0|\vec{u}) + D(u_0|\vec{u})$ on the state $|M\rangle$, one needs the commutation relations between the fields A, B, D. From (2.18) one reads

$$
\begin{aligned}
A(u_0|\vec{u})B(w|\vec{u}) &= \frac{1}{b(w,u_0)} B(w|\vec{u})A(u_0|\vec{u}) - \frac{a(w,u_0)}{b(w,u_0)} B(u_0|\vec{u})A(w|\vec{u}) \\
B(w_1|\vec{u})B(w_2|\vec{u}) &= B(w_2|\vec{u})B(w_1|\vec{u}) \\
D(u_0|\vec{u})B(w|\vec{u}) &= \frac{1}{b(u_0,w)} B(w|\vec{u})D(u_0|\vec{u}) - \frac{a(u_0,w)}{b(u_0,w)} B(u_0|\vec{u})D(w|\vec{u}).
\end{aligned}
\tag{B.10}
$$

From this, one can compute exactly when $|M\rangle$ is an eigenstate of the transfer matrix. By definition we have that

$$|M\rangle = B(w_M|\vec{u})|M-1\rangle, \tag{B.11}$$

and this allows us to use induction. By using the identity

$$\frac{1}{b(w_M,u_0)} \frac{a(w_i,u_0)}{b(w_i,u_0)} - \frac{a(w_M,u_0)}{b(w_M u_0)} \frac{a(w_i,w_M)}{b(w_i,w_M)} = \frac{a(w_i,u_0)}{b(w_i,u_0)} \frac{1}{b(w_M,w_i)} \tag{B.12}$$

in (B.11) one can prove

$$
\begin{aligned}
A(u_0|\vec{u})\phi_M(w_1, \ldots, w_M) &= \prod_{i=1}^{M} \frac{1}{b(w_i,u_0)} \phi_M(w_1, \ldots, w_M)A(u_0|\vec{u}) \\
&\quad - \sum_{i=1}^{M} \left[\frac{a(w_i,u_0)}{b(w_i,u_0)} \prod_{j=1,j\neq i}^{M} \frac{1}{b(w_j,w_i)} \hat{\phi}_M A(w_i|\vec{u}) \right],
\end{aligned}
\tag{B.13}
$$

where $\hat{\phi}_M$ stands for $\phi_M(\ldots, w_{i-1}, u_0, w_{i+1}, \ldots)$. One can find a similar relation for the commutator between D and B. By using these relations one finds that

$$
\begin{aligned}
\mathscr{T}(u_0|\vec{u})|M\rangle &= \{A(u_0|\vec{u}) + D(u_0|\vec{u})\}|M\rangle \tag{B.14} \\
&= \phi_M(w_1, \ldots, w_M) \left\{ A(u_0|\vec{u}) \prod_{i=1}^{M} \frac{1}{b(w_i,u_0)} + D(u_0|\vec{u}) \prod_{i=1}^{M} \frac{1}{b(u_0,w_i)} \right\} |0\rangle \\
&\quad - \sum_{i=1}^{M} \left[\frac{a(w_i,u_0)}{b(w_i,u_0)} \hat{\phi}_M \left\{ \prod_{j\neq i} \frac{1}{b(w_j,w_i)} A(w_i|\vec{u}) - \prod_{j\neq i} \frac{1}{b(w_i,w_j)} D(w_i|\vec{u}) \right\} \right] |0\rangle.
\end{aligned}
$$

From this we find that $|M\rangle$ is an eigenstate of the transfer matrix with eigenvalue

$$\Lambda^{(6v)}(u_0|\vec{u}) = \prod_{i=1}^{M} \frac{1}{b(w_i, u_0)} + \prod_{i=1}^{M} \frac{1}{b(u_0, w_i)} \prod_{i=1}^{K} b(u_0, u_i) \tag{B.15}$$

provided that the auxiliary parameters w_i satisfy the following equations

$$\prod_{i=1}^{K} b(w_j, u_i) = \prod_{i=1, i \neq j}^{M} \frac{b(w_j, w_i)}{b(w_i, w_j)}. \tag{B.16}$$

This now completely determines the spectrum of the 6-vertex model.

To conclude, we briefly explain how these eigenvalues are used to generate an infinite tower of conserved charges. From (2.18) one finds that

$$\mathscr{T}(u_0|\vec{u})\mathscr{T}(\mu|\vec{u}) = \mathscr{T}(\mu|\vec{u})\mathscr{T}(u_0|\vec{u}). \tag{B.17}$$

This means that if one writes $\mathscr{T}(u_0|\vec{u})$ as a series the auxiliary parameter u_0, the coefficients of this series will depend on $\vec{u}$ and are in involution with each other. It actually turns out that the 6-vertex model Hamiltonian can be written in terms of these coefficients and hence this means that the coefficients of $\mathscr{T}(u_0|\vec{u})$ yield an infinite number of conserved quantities.

C Alternative vacua

In this section we will discuss a class of higher excited states for which we present a full derivation of its eigenvalue of the transfer matrix and the auxiliary Bethe equations. From the general construction it is easily seen that a more general eigenvector of the transfer matrix is given by

$$|a\rangle = \Phi(\lambda_1, \ldots, \lambda_a)|0\rangle_P, \qquad \Phi(\lambda_1, \ldots, \lambda_a) = B_3(\lambda_1) \ldots B_3(\lambda_a). \tag{C.1}$$

These states have quantum number $K^{\text{III}} = 0$. This allows for a similar inductive procedure as applied to the 6-vertex model in section B. Furthermore, because of the properties of the creation operators (3.32), we find that

$$\Phi(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \ldots \lambda_a) = -\mathscr{X}_0^{0,0} r_{33}^{33}(\lambda_{j-1}, \lambda_j) \Phi(\lambda_1, \ldots, \lambda_j, \lambda_{j-1}, \ldots \lambda_a). \tag{C.2}$$

This means that all permutations of the momenta λ_i are related to each other by a simple multiplication by a scalar prefactor. We will exploit this property later on. Let us first derive some useful identities. One uses induction to show that

$$A_3^4|a\rangle = C_3^*|a\rangle = C_4|a\rangle = C|a\rangle = 0, \tag{C.3}$$

for any a. This vastly simplifies the computations, since we can discard any term proportional to the above operators from the commutation relations. Let us first turn to (3.30). This now becomes, after discarding the term proportional to C_3^*,

$$\mathcal{T}_{3,k}^{3,k}(q)B_3(\lambda) = \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} B_3(\lambda)\mathcal{T}_{3,k}^{3,k}(q) + \frac{\mathscr{Y}_{k;2}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^{k}(q)A_3^3(\lambda) + \frac{\mathscr{Y}_{k;4}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda).$$

Applying this to $\Phi(\lambda_1, \ldots, \lambda_a) = B_3(\lambda_1)\Phi(\lambda_2, \ldots, \lambda_a)$ we find

$$
\begin{aligned}
\mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \ldots, \lambda_a) = {} & \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} B_3(\lambda_1)\mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_2, \ldots, \lambda_a) \\
& + \frac{\mathscr{Y}_{k;2}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^k(q)A_3^3(\lambda_1)\Phi(\lambda_2, \ldots, \lambda_a) \\
& + \frac{\mathscr{Y}_{k;4}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\mathcal{T}_{3,k}^{34,k-1}(q)A_3^3(\lambda_1)\Phi(\lambda_2, \ldots, \lambda_a).
\end{aligned} \tag{C.4}
$$

Obviously, by applying this relation recursively one finds

$$
\begin{aligned}
\mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \ldots, \lambda_a) = {} & \prod_{i=1}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}\Phi(\lambda_1, \ldots, \lambda_a)\mathcal{T}_{3,k}^{3,k}(q) \\
& + \sum_{i=1}^{a} c_i \Phi_{k;i}(q,\lambda)A_3^3(\lambda_i) + \sum_{i=1}^{a} d_i \Psi_{k;i}(q,\lambda)A_3^3(\lambda_i),
\end{aligned} \tag{C.5}
$$

where c_i are some numerical coefficients and $\Phi_{k;i}(q,\lambda) = T_{3,k}^k(q)\prod_{j\neq i} B_3(\lambda_j), \Psi_{k;i}(q,\lambda) = T_{3,k}^{34,k-1}(q)\prod_{j\neq i} B_3(\lambda_j$ It is easily seen from (C.4) that the numerical coefficients in front of $\Phi_{k;1}(q,\lambda), \Psi_{k;1}(q,\lambda)$ are given by

$$
c_1 = \frac{\mathscr{Y}_{k;2}^{k,1;1}(q,\lambda_1)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_1)}\prod_{i=2}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}, \qquad d_1 = \frac{\mathscr{Y}_{k;4}^{k,1;1}(q,\lambda_1)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_1)}\prod_{i=2}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}. \tag{C.6}
$$

Here we can exploit the symmetry property (C.2) to relate all the other coefficients to this one. Let us denote these proportionality coefficients by $\mathcal{P}_{1i}$, then we find

$$
\begin{aligned}
\mathcal{T}_{3,k}^{3,k}(q)\Phi(\lambda_1, \ldots, \lambda_a) = {} & \prod_{i=1}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}\Phi(\lambda_1, \ldots, \lambda_a)\mathcal{T}_{3,k}^{3,k}(q) \\
& + \sum_{i=1}^{a} c_i P_{1i}\Phi_{k;i}(q,\lambda)A_3^3(\lambda_i) + \sum_{i=1}^{a} d_i P_{1i}\Psi_{k;i}(q,\lambda)A_3^3(\lambda_i),
\end{aligned} \tag{C.7}
$$

where

$$
c_j = \frac{\mathscr{Y}_{k;2}^{k,1;1}(q,\lambda_j)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_j)}\prod_{i=1,i\neq j}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}, \qquad d_j = \frac{\mathscr{Y}_{k;4}^{k,1;1}(q,\lambda_j)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_j)}\prod_{i=1,i\neq j}^{a} \frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}. \tag{C.8}
$$

Next, we consider the commutator with $\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}$. Upon dismissing vanishing terms we find

$$
\begin{aligned}
\left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}\right] B_3(\lambda) = {} & \frac{\mathscr{X}_k^{k,0}}{\mathscr{Y}_{k;1}^{k,0;1}} B_3(\lambda)\left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}\right] + \\
& \frac{\mathscr{Y}_{k;2}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\left\{\mathcal{T}_{3,k}^k B - \mathcal{T}_{34,k-1}^{4,k-1}A_3^3\right\} + \frac{\mathscr{Y}_{k;4}^{k,1;1}}{\mathscr{Y}_{k;1}^{k,0;1}}\left\{\mathcal{T}_{3,k}^{34,k-1} B + \mathcal{T}_k^{4,k-1}A_3^3\right\}.
\end{aligned} \tag{C.9}
$$

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If we now define $\hat{\Phi}_{k;i}(q,\lambda) = \mathcal{T}_k^{4,k-1}(q)\prod_{j\neq i} B_3(\lambda_j)$, $\hat{\Psi}_{k;i}(q,\lambda) = \mathcal{T}_{34,k-1}^{4,k-1}(q)\prod_{j\neq i} B_3(\lambda_j)$, then we can repeat the above steps to find

$$
\begin{aligned}
\left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}\right]\Phi(\lambda_1,\ldots,\lambda_a) &= \prod_{i=1}^{a}\frac{\mathscr{X}_k^{k,0}(q,\lambda_i)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_i)}\Phi(\lambda_1,\ldots,\lambda_a)\left[\mathcal{T}_k^k + \mathcal{T}_{34,k-1}^{34,k-1}\right] + \\
&\quad + \sum_{i=1}^{a} c_i P_{1i}\left\{\Phi_{k;i}(q,\lambda)B(\lambda_i) - \hat{\Psi}_{k;i}(q,\lambda)A_3^3(\lambda_i)\right\} + \quad \text{(C.10)} \\
&\quad + \sum_{i=1}^{a} d_i P_{1i}\left\{\Psi_{k;i}(q,\lambda)B(\lambda_i) + \hat{\Phi}_{k;i}(q,\lambda)A_3^3(\lambda_i)\right\}.
\end{aligned}
$$

The last commutation relation finally gives

$$
\begin{aligned}
\mathcal{T}_{4,k}^{4,k}(q)\Phi(\lambda_1,\ldots,\lambda_a) &= \frac{\mathscr{X}_{k+1}^{k+1,0}}{\mathscr{Y}_{k;1}^{k,0;1}}\frac{u_q - u_{\lambda_1} + \frac{\ell_0-1}{2} - k}{u_q - u_{\lambda_1} + \frac{\ell_0-3}{2} - k}B_3(\lambda_1)\mathcal{T}_{4,k}^{4,k}(q)\Phi(\lambda_2,\ldots,\lambda_a) \\
&\quad - \frac{\mathscr{Y}_{k+1;2}^{k+1,1;1}}{\mathscr{Y}_{k+1;1}^{k+1,0;1}}\mathcal{T}_{34,k}^{4,k}(q)B(\lambda_1)\Phi(\lambda_2,\ldots,\lambda_a) \quad \text{(C.11)} \\
&\quad + \frac{\mathscr{Y}_{k+1;4}^{k+1,1;1}}{\mathscr{Y}_{k+1;1}^{k+1,0;1}}\mathcal{T}_{k+1}^{4,k}(q)B(\lambda_1)\Phi(\lambda_2,\ldots,\lambda_a).
\end{aligned}
$$

Again the same arguments apply as above.

By summing now all the terms, we find that $|a\rangle$ is indeed an eigenstate of the transfer matrix, provided that the parameters λ_i satisfy

$$
B(\lambda_i)|0\rangle_P = A_3^3(\lambda_i)|0\rangle_P. \tag{C.12}
$$

When working this out, we see that the above only depends on $x^+(\lambda_i)$, which we denote as $y_i \equiv x^+(\lambda_i)$. The explicit formula is given by

$$
\prod_{j=1}^{K^{\mathrm{I}}}\frac{y_i - x_j^+}{y_i - x_j^-}\sqrt{\frac{x_j^-}{x_j^+}} = 1, \tag{C.13}
$$

which agrees with the known auxiliary BAE. The explicit eigenvalue of $|a\rangle$ is given by

$$
\begin{aligned}
\Lambda(q|\vec{p}) &= \prod_{m=1}^{K^{\mathrm{II}}}\frac{\mathscr{X}_0^{0,0}(q,\lambda_m)}{\mathscr{Y}_{0;1}^{0,0;1}(q,\lambda_m)} + \prod_{i=1}^{K^{\mathrm{I}}}\mathscr{L}_{\ell_0;1}^{\ell_0,0;1}(q,p_i)\prod_{m=1}^{K^{\mathrm{II}}}\frac{\mathscr{X}_{\ell_0}^{\ell_0,0}(q,\lambda_m)}{\mathscr{Y}_{\ell_0;1}^{\ell_0,0;1}(q,\lambda_m)} + \\
&\quad \sum_{k=1}^{\ell_0-1}\prod_{m=1}^{K^{\mathrm{II}}}\frac{\mathscr{X}_k^{k,0}(q,\lambda_m)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_m)}\left\{\prod_{i=1}^{K^{\mathrm{I}}}\lambda_+(q,p_i) + \prod_{i=1}^{K^{\mathrm{I}}}\lambda_-(q,p_i)\right\} + \quad \text{(C.14)} \\
&\quad - \sum_{k=0}^{\ell_0-1}\prod_{m=1}^{K^{\mathrm{II}}}\frac{\mathscr{X}_k^{k,0}(q,\lambda_m)}{\mathscr{Y}_{k;1}^{k,0;1}(q,\lambda_m)}\left[1 + \frac{u_q - u_{\lambda_m} + \frac{\ell_0}{2} - k}{u_q - u_{\lambda_m} + \frac{\ell_0-2}{2} - k}\right]\prod_{i=1}^{K^{\mathrm{I}}}\mathscr{Y}_{k;1}^{k,0;1}(q,p_i).
\end{aligned}
$$

This is indeed the case $K^{\mathrm{III}} = 0$ of (3.45).

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